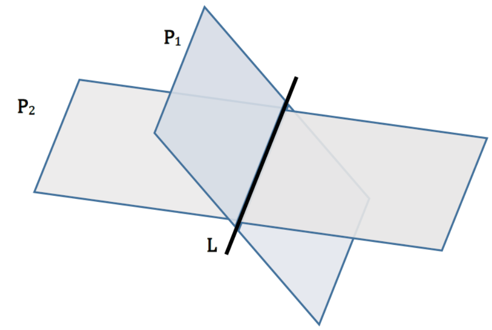
V9 Intersecting Planes



Introduction

When two planes in three dimensions intersect, the set of points in the intersection forms a line. However two planes do not form lines if they are parallel and hence never intersect.

This module considers the angle between two planes when they are not parallel.

Definition of a Plane

A plane in three dimensions is a set of points satisfying the equation

$$ax + by + cz = d$$

where a, b, c and d are constants.

Normal to a Plane

The normal $\vec{N}$, of a plane

$$ax + by + cz = d$$

is a vector at right angles to the plane and has the form

$$\vec{N} = a\hat{i} + b\hat{j} + c\hat{k}.$$

Of course, another normal is given by $-\vec{N} = -a\hat{i} - b\hat{j} - c\hat{k}$ as this just a vecor in the opposite direction of $\vec{N}$ and is still at right angles to the plane.

The Angle Between Two Planes

The angle between two intersecting planes is the same as the angle between the normals of the planes.

If N_1 and $N_2$are the normals of two intersecting planes , we know that [1]

[1] This follows from the definition of the dot or scalar product.

$$\overrightarrow{N_1}.\overrightarrow{N_2} = \left|\overrightarrow{N_1}\right| \left|\overrightarrow{N_2}\right| \cos\theta$$

where θ is the angle between them.

Rearranging , we find :

$$\cos\theta = \frac{\overrightarrow{N_1}.\overrightarrow{N_2}}{\left|\overrightarrow{N_1}\right| \left|\overrightarrow{N_2}\right|}.$$

Note that there are in general two possible angles. One is obtuse, the other is acute. Together they sum to 180°.

If we wish to find only the acute angle between the planes then we say that:

$$\cos\theta = \frac{\left|\overrightarrow{N_1}.\overrightarrow{N_2}\right|}{\left|\overrightarrow{N_1}\right| \left|\overrightarrow{N_2}\right|}$$

Example

Find the acute angle of intersection of the planes $x + y + z = 0$ and $x - 3y + z = 1$.

The plane $x + y + z = 0$ has the normal vector $\overrightarrow{N_1} = \overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$.

The plane $x - 3y + z = 1$ has the normal vector $\overrightarrow{N_2} = \overrightarrow{i} - 3\overrightarrow{j} + \overrightarrow{k}$.

Remember:

$$\cos\theta = \frac{\left|\overrightarrow{N_1}.\overrightarrow{N_2}\right|}{\left|\overrightarrow{N_1}\right| \left|\overrightarrow{N_2}\right|}$$

so:

$$\begin{aligned}\left|\overrightarrow{N_1}.\overrightarrow{N_2}\right| &= |(1 \times 1) + (1 \times -3) + (1 \times 1)| \\ &= |-1| \\ &= 1\end{aligned}$$

and:

$$\begin{aligned}\left|\overrightarrow{N_1}\right| &= \sqrt{1^2 + 1^2 + 1^2} \\ &= \sqrt{3}\end{aligned}$$

and:

$$\left|\overrightarrow{N_2}\right| = \sqrt{1^2 + 3^2 + 1^2}$$
$$= \sqrt{11}$$

Therefore:

$$\cos\theta = \frac{1}{\sqrt{3}\sqrt{11}} \simeq 0.1741$$

Rearranging formula and using inverse cos gives:

$$\theta = \cos^{-1}(0.1741) \simeq 80^o$$

So the angle of intersection of the two planes is approximately 80 degrees.

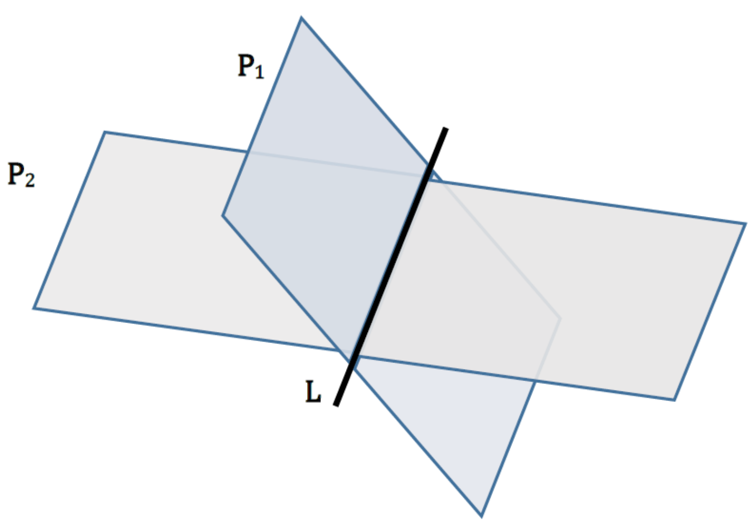
Exercise

Find the angle of intersection of the following planes

1. The plane $x - y + z = 1$ and $2x + y - z = 3$
   Answer: 90 degrees.
2. The plane $2x + y - z = 2$ and $3x + y - z = 3$
   Answer: 10 degrees.

Line of Intersection of Two Planes

The planes, $P_1$and P_2 intersect along the line L, as in the diagram below:



We want to find the equation of the line of intersection, L.

Example

If P_1: $2x + 4y - z = 4$ and P_2: $x - 2y + z = 3$, find the parametric equations of the line of intersection of the two planes.

Solution:

Given $2x + 4y - z = 4$ and $x - 2y + z = 3$, we have two equations but three unknowns. This is a clue to introduce a parameter[2]. Let $z = t$ then the equations of the planes become

[2] We will set $z = t$ but you can set $x = t$ or $y = t$. This will generate a set of equations that may look different to what we show below, but they are correct.

$$
\begin{aligned}
2x + 4y - t &= 4 \ (1) \\
x - 2y + t &= 3.(2)
\end{aligned}
$$

Multiplying (2) by -2, the equations become:

$$
\begin{aligned}
2x + 4y - t &= 4 \\
-2x + 4y - 2t &= -6
\end{aligned}
$$

adding these two equations we get:

$$
\begin{aligned}
8y - 3t &= -2 \\
8y &= 3t - 2 \\
y &= \frac{3}{8}t - \frac{1}{4}.
\end{aligned}
$$

Substituting y in equation (2)

$$
\begin{aligned}
x - 2\left(\frac{3}{8}t - \frac{1}{4}\right) + t &= 3 \\
x - \frac{6}{8}t + \frac{2}{4} + t &= 3 \\
x + \frac{1}{4}t + \frac{1}{2} &= 3 \\
x &= 3 - \frac{1}{4}t - \frac{1}{2} \\
x &= \frac{5}{2} - \frac{1}{4}t.
\end{aligned}
$$

Hence the parametric equations of the line of the intersection of the two planes are:

$$x = \frac{5}{2} - \frac{1}{4}t, \ y = \frac{3}{8}t - \frac{1}{4} \text{ and } z = t.$$

Exercise

Find the parametric line of intersection and the angle of intersection of the planes $x + y + 2z = 0$ and $2x - y + z = 5$.

Answer: Line is $x = 2 - t$; $y = -1 - t$; $z = t$. Angle is 60degrees.