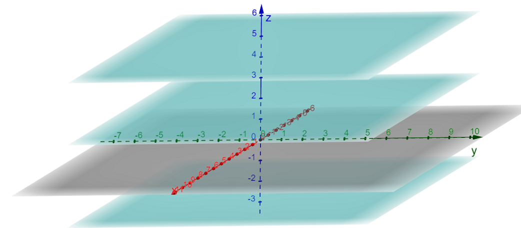
V8 Equation of a Plane

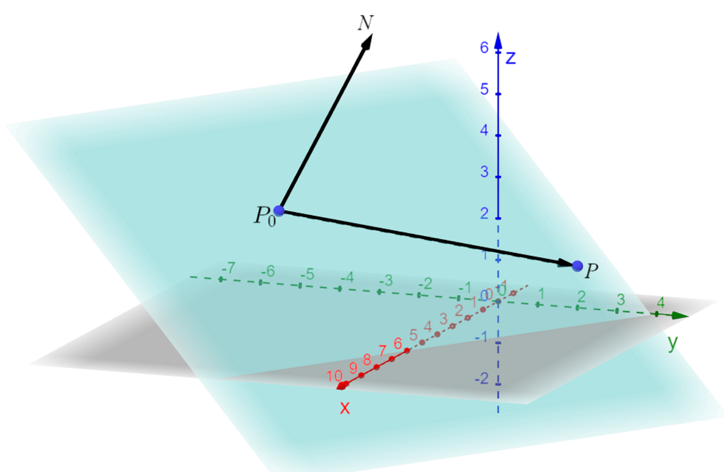


A plane is a subset of three dimensional space. In non mathematical terms you can think of it as a flat surface that extends infinitely in two directions. It may be defined as

1. the surface that goes through three points or

2. the surface containing a point and having a fixed normal vector.

Cartesian Equation of a Plane



The above diagram represents a part of a plane. On the plane are two points, $P(x, y, z)$ and $P_0(x_0, y_0, z_0)$.

The vector $\overrightarrow{P_0P} = (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}$.

Also illustrated is a normal vector to the plane (that is a vector at right angles to the plane) represented by $\overrightarrow{N} = a\hat{i} + b\hat{j} + c\hat{k}$, where a, b and c are constants.

Since $\overrightarrow{N}$ is perpendicular to $\overrightarrow{P_0P}$ their dot product must equal zero[1], that is to say $\overrightarrow{N}.\overrightarrow{P_0P} = 0$ therefore:

[1] The definition of the dot product of two vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. If $\vec{a}$ and $\vec{b}$ are perpendicular, $\theta = 90°$. Hence $\cos(\theta) = \cos(90°) = 0$ and so

$$\vec{a} \cdot \vec{b} = 0.$$

$$\left(a\hat{\imath}+b\hat{\jmath}+c\hat{k}\right).\left((x-x_0)\hat{\imath}+(y-y_0)\hat{\jmath}+(z-z_0)\hat{k}\right)=0$$

So

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

This equation defines the plane in Cartesian coordinates.

This equation may be rearranged:

$$\begin{aligned} a(x-x_0)+b(y-y_0)+c(z-z_0)&=0\\ ax+by+cz&=ax_0+by_0+cz_0\\ &=d \end{aligned}$$

where d is a constant. In fact, the general equation of a plane with the normal vector $\overrightarrow{N}=a\hat{\imath}+b\hat{\jmath}+c\hat{k}$ is

$$ax+by+cz=d.$$

For example,

1. $2x-3y+z=6$ is a plane.
2. $x+y=4$ is a plane.
3. $z=x-2y+3$ is a plane. To see this rearrange it to get the Cartesian form of the plane, $x-2y-z=-3$.

Example 1

If a plane has the normal vector $\overrightarrow{N}=\hat{\imath}+2\hat{\jmath}-5\hat{k}$ and contains the point $(3,4,1)$ then we can say that the equation of the plane is:

$$\begin{aligned} 1(x-3)+2(y-4)-5(z-1)&=0\\ x-3+2y-8-5z+5&=0\\ x+2y-5z&=3+8-5\\ x+2y-5z&=6 \end{aligned}$$

Compare this equation with the normal vector, $\overrightarrow{N}$. You will notice that the coefficients of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ for the normal vector are the same as the coefficients of x, y and z in the equation of the plane.

Example 2

Find the equation of the plane normal to the vector $6\hat{\imath}-5\hat{\jmath}+\hat{k}$ that passes through the point $(1,2,3)$.

The equation of the plane will be [2]

[2] Obviously there are an infinite number of parallel planes that are perpendicular to the given vector. For instance $6x-5y+z=-1$, $6x-5y+z=0$, $6x-5y+z=6$, $6x-5y+z=11$ etc. are all perpendicular to the vector $6\hat{\imath}-5\hat{\jmath}+\hat{k}$ but only one of these, $6x-5y+z=-1$, will pass through the given point.

$$\begin{aligned} 6(x-1)-5(y-2)+1(z-3)&=0 \\ 6x-6-5y+10+z-3&=0 \\ 6x-5y+z&=6-10+3 \\ 6x-5y+z&=-1 \end{aligned}$$

Not only can we find the equation of a plane, given a point and a normal vector, but we can also find the equation of the normal vector given the equation of a plane.

For example the plane $3x+4y-z=9$ has a normal vector $\overrightarrow{N}=3\hat{\imath}+4\hat{\jmath}-\hat{k}$.

Example 3

Find the equation of the plane that passes through the point $(2,1,-5)$ and is parallel to the plane $z=2x+3y-4$.

Since the plane is parallel to $z=2x+3y-4$, it will have the same normal vector.

The equation $z=2x+3y-4$ can be written as $2x+3y-z=4$, therefore its normal vector will be $2\hat{\imath}+3\hat{\jmath}-\hat{k}$.

So the equation of the plane we wish to find will pass through the point $(2,1,-5)$ and have a normal vector $2\hat{\imath}+3\hat{\jmath}-\hat{k}$.

$$\begin{aligned} 2(x-2)+3(y-1)-1(z+5)&=0 \\ 2x-4+3y-3-z-5&=0 \\ 2x+3y-z&=4+3+5 \\ 2x+3y-z&=12 \end{aligned}$$

Example 4

Find the equation of the plane that contains the three points $A(1,1,1)$, $B(2,4,3)$ and $C(3,2,1)$.

First find the cross product of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as this will give a vector normal to the plane.

Then use this normal vector and any one of the three given points to find the equation of the plane.

$\overrightarrow{AB}\times\overrightarrow{AC}=(1\hat{\imath}+3\hat{\jmath}+2\hat{k})\times(2\hat{\imath}+\hat{\jmath})$

$$=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & 2 \\ 2 & 1 & 0 \end{vmatrix}=-2\hat{\imath}+4\hat{\jmath}-5\hat{k}$$

So the plane has the normal vector $\overrightarrow{N}=-2\hat{\imath}+4\hat{\jmath}-5\hat{k}$ and contains the point $(1,1,1)$[3]

[3] Either of the other two points could also have been used

Therefore the equation of the plane is

$$\begin{aligned}
-2(x-1)+4(y-1)-5(z-1) &= 0 \\
-2x+2+4y-4-5z+5 &= 0 \\
-2x+4y-5z &= -2+4-5 \\
-2x+4y-5z &= -3 \\
\text{or } 2x-4y+5z &= 3.
\end{aligned}$$

Exercises

1. What are the vectors normal to the following planes,
   (a) $2x+3y+7z=13$
   (b) $z=x+3y-9$.
   Answer: a) $2\hat{i}+3\hat{j}+7\hat{k}$ b) $\hat{i}+3\hat{j}-\hat{k}$.
2. Find the equation of the plane that contains the point $(2,3,1)$ and has the normal vector $4\hat{i}+3\hat{j}-2\hat{k}$.
   Answer: $4x+3y-2z=15$.
3. Find the equation of the plane that contains the point $(5,-3,2)$ and has the normal vector $\hat{i}-9\hat{j}-4\hat{k}$.
   Answer: $x-9y-4z=24$.
4. Find the equation of the plane that contains the point $(1,-1,0)$ and is parallel to the plane $x-3y+2z=0$.
   Answer: $x-3y+2z=4$.
5. Find the equation of the plane that contains the points $(1,2,3)$, $(2,1,1)$ and $(-3,0,4)$.
   Answer: $5x-7y+6z=9$.