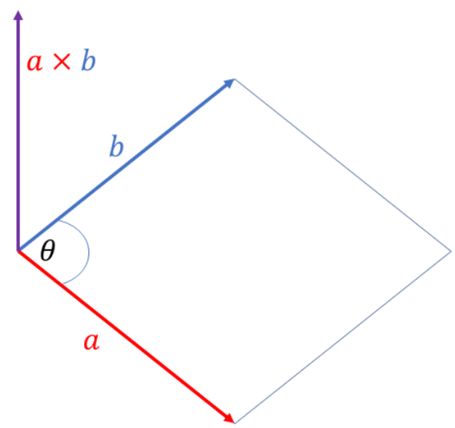
V4: Vector Product



There are two ways to multiply two vectors:

1. The scalar product which gives a number (also called the dot product);
2. The vector product which gives a vector (also called the cross product).

In this module we consider the vector or cross product.

Definition

Let $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ be unit vectors in the x, y and z directions respectively. We can write

$$\begin{aligned}\hat{\imath} &= \hat{\imath} + 0\hat{\jmath} + 0\hat{k} \\ &= (1, 0, 0) \\ \hat{\jmath} &= 0\hat{\imath} + \hat{\jmath} + 0\hat{k} \\ &= (0, 1, 0) \\ \hat{k} &= 0\hat{\imath} + 0\hat{\jmath} + \hat{k} \\ &= (0, 0, 1)\,.\end{aligned}$$

Let the vectors

$$\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

and

$$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}.$$

The vector, or cross, product of the two vectors $\vec{a}$ and $\vec{b}$ is the vector[1]

$$\vec{a} \times \vec{b} = |\vec{a}|\left|\vec{b}\right| \sin(\theta)\ \hat{n} \qquad (1)$$

where $\hat{n}$ is a unit vector that is perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

[1] Here, $|\vec{a}|$ and $\left|\vec{b}\right|$ are the magnitudes of the vectors $\vec{a}$ and $\vec{b}$.

To calculate the cross product, it is more convenient to use the definition[2]

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$
$$= \hat{i}\left(a_2b_3 - b_2a_3\right) - \hat{j}\left(a_1b_3 - b_1a_3\right) + \hat{k}\left(a_1b_2 - b_1a_2\right). \qquad (2)$$

[2] Here the two vertical delimiters $|\mathbf{A}|$ denote the determinant of the matrix $\mathbf{A}$. For a two by two matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the determinant

$$|\mathbf{A}| = ad - bc.$$

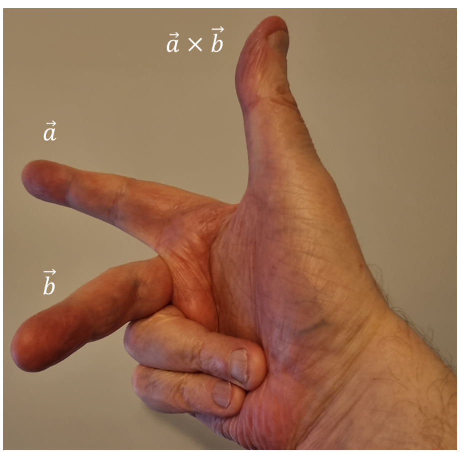
This is quite different to the magnitude of a vector $|\vec{a}|$.

Properties of the Vector or Cross Product

1. If $\vec{a}$ is parallel to $\vec{b}$ then $\vec{a}\times\vec{b} = \vec{0}$. This follows from eqn (1) above[3]. Note that we should write the answer as the zero vector $\vec{0}$ instead of the number 0.[4]

2. The order in which you take the cross product is important. In fact, $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$.

3. The direction of $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ in the direction in which your thumb would point if the fingers of your right hand are curled from $\vec{a}$ to $\vec{b}$ as shown below



This is called the right hand rule.

[3] If $\vec{a}$ is parallel to $\vec{b}$ the angle between them is 0. Hence from eqn(1)

$$\vec{a}\times\vec{b} = |\vec{a}|\left|\vec{b}\right|\sin(0)\ \hat{n}$$
$$= \vec{0}$$

as $\sin(0) = 0$.

[4] This is a technical point and relates to the concept of a vector space in which certain operations are closed. Closed means if you do operations on a vector you get a vector as a result. In many courses you don't need to worry about this so writing $\vec{a}\times\vec{b} = 0$ may be allowed. Please see you teacher on this point.

Example 1

Find $\vec{a}\times\vec{b}$ if $\vec{a} = 2\vec{i} + 3\vec{j} + \vec{k}$ and $\vec{b} = 5\vec{j} + 3\vec{k}$.

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 1 \\ 0 & 5 & 3 \end{vmatrix}$$
$$= \hat{\imath}\,(3 \times 3 - 5 \times 1) - \hat{\jmath}\,(2 \times 3 - 0 \times 1) + \hat{k}\,(2 \times 5 - 0 \times 3)$$
$$= (9 - 5)\,\hat{\imath} - (6 - 0)\,\hat{\jmath} + (10 - 0)\,\hat{k}$$
$$= 4\hat{\imath} - 6\hat{\jmath} + 10\hat{k}.$$

Example 2

Find $\vec{a} \times \vec{b}$ if $\vec{a} = (2, 1, 1)$ and $\vec{b} = (-2, 4, 0)$.

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 1 \\ -2 & 4 & 0 \end{vmatrix}$$
$$= \hat{\imath}\,(1 \times 0 - 4 \times 1) - \hat{\jmath}\,(2 \times 0 - (-2) \times 1) + \hat{k}\,(2 \times 4 - (-2) \times 1)$$
$$= -4\hat{\imath} - 2\hat{\jmath} + 10\hat{k}.$$

Example 3

Find $\vec{a} \times \vec{b}$ if $\vec{a} = (2, 1, 1)$ and $\vec{b} = (8, 4, 4)$.

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{vmatrix}$$
$$= \hat{\imath}\,(1 \times 4 - 4 \times 1) - \hat{\jmath}\,(2 \times 4 - 8 \times 1) + \hat{k}\,(2 \times 4 - 8 \times 1)$$
$$= 0\hat{\imath} - 0\hat{\jmath} + 0\hat{k}$$
$$= 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$
$$= (0, 0, 0)$$
$$= \vec{0}.$$

Note that $0\hat{\imath} - 0\hat{\jmath} + 0\hat{k} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$ and that the answer is the **vector** $(0, 0, 0) = \vec{0}$, not simply the number 0 (see footnote 2 above).

Since neither$|\vec{a}|$ nor $\left|\vec{b}\right|$ is zero, from equation (1) above, we can see that $\sin(\theta) = 0$ and so $\theta = 0$ or $\theta = \pi$. That is the vectors $\vec{a}$ and $\vec{b}$ are in the same or opposite directions. This result could be more quickly obtained by observing that $\vec{b} = 4\vec{a}$ and so $\vec{a}$ and $\vec{b}$ are parallel and by property (1) of the vector product above, $\vec{a} \times \vec{b} = \vec{0}$.

Cartesian Unit Vectors

Let $\hat{i}, \hat{j}$ and $\hat{k}$ be unit vectors in the x, y and z directions respectively. Then

$$\begin{aligned} \hat{i} \times \hat{j} &= \hat{k} & \hat{j} \times \hat{k} &= \hat{i} & \hat{k} \times \hat{i} &= \hat{j} \\ \hat{i} \times \hat{k} &= -\hat{j} & \hat{k} \times \hat{j} &= -\hat{i} & \hat{j} \times \hat{i} &= -\hat{k}. \end{aligned}$$

These results may be confirmed with the right hand rule or equation (2) above.

Example 4

Find $\hat{i} \times \hat{k}$.

Solution:

We have

$$\begin{aligned} \hat{i} &= \hat{i} + 0\hat{j} + 0\hat{k} \\ \hat{k} &= 0\hat{i} + 0\hat{j} + \hat{k}. \end{aligned}$$

Using equation (2) ,

$$\begin{aligned} \hat{i} \times \hat{k} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \hat{i}\,(1 \times 0 - 0 \times 0) - \hat{j}\,(1 \times 1 - 0 \times 0) + \hat{k}\,(1 \times 0 - 0 \times 0) \\ &= -\hat{j}. \end{aligned}$$

Exercise 1

Calculate the following.

1. $\hat{j} \times \hat{k}$.
2. $\hat{i} \times 4\hat{i}$.
3. $\left(2\hat{i} + 3\hat{j} - \hat{k}\right) \times \left(3\hat{j} + 2\hat{k}\right)$.
4. $3\hat{j} \times 5\hat{i}$.
5. $\left(\hat{i} - 3\hat{j} + \hat{k}\right) \times \left(2\hat{i} + \hat{j} - \hat{k}\right)$.

Answers

1. $\hat{i}$ 2. $\hat{0}$ 3. $9\hat{i} - 4\hat{j} + 6\hat{k}$ 4. $-15\hat{k}$ 5. $\left(2\hat{i} + 3\hat{j} + 7\hat{k}\right)$.

Exercise 2

Find a unit vector perpendicular to both $\left(\vec{i} - \vec{k}\right)$ and $\left(\vec{i} + 3\vec{j} - 2\vec{k}\right)$.

Answer

$\frac{(3,1,3)}{\sqrt{19}}$ or $-\frac{(3,1,3)}{\sqrt{19}}$.