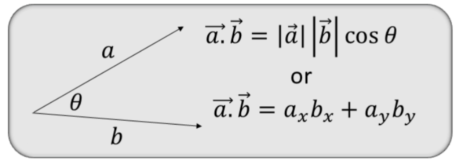
V3: Scalar Product



There are two ways to multiply two vectors:

1. The scalar or dot product which gives a number;
2. The vector or cross product which gives a vector.

In this module we consider the scalar or dot product.

Definition

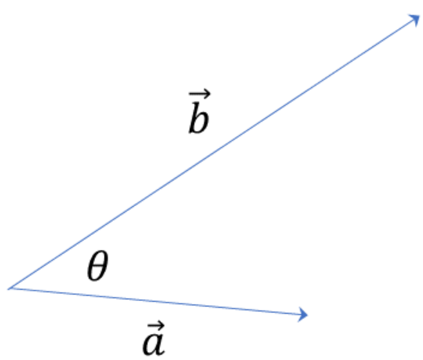
The scalar, or dot, product of two vectors $\vec{a}\,(a_1, a_2, a_3)$ and $\vec{b}\,(b_1, b_2, b_3)$ is a scalar, defined by:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

or geometrically,

$$\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.



Properties of the Scalar or Dot Product

1. If $\vec{a}$ and $\vec{b}$ are non-zero vectors and $\vec{a}$ is perpendicular[1] to $\vec{b}$ then $\vec{a} \cdot \vec{b} = 0$, since $\cos\left(\frac{\pi}{2}\right) = 0$.
2. If $\vec{a}$ is parallel to $\vec{b}$ then the angle between the vectors is 0 and $\vec{a} \cdot \vec{b} = \mid \vec{a} \mid\mid \vec{b} \mid$ as $\cos(0) = 1$.

[1] Perpendicular means at right angles to. A right angle is $90^\circ = \pi/2$.

3. The dot product does not depend on the order of multiplication:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

4. In three dimensions with $\hat{\imath}$,$\hat{\jmath}$ and $\hat{k}$ unit vectors along the x, y and z axes respectively, we have:

$$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$$
$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$$

Examples

1. $\left(2\vec{i} + 3\vec{j} + 4\vec{k}\right) \cdot \left(-\vec{i} - 2\vec{j} + \vec{k}\right) = (2 \times (-1)) + (3 \times (-2)) + (4 \times 1) = -4$

2. $(2, -3, -3) \cdot (1, 1, -2) = 2 - 3 + 6 = 5$

3. $(5, 0, -1) \cdot (1, 4, 3) = 5 + 0 - 3 = 2$

4. $\left(2\vec{i} + 4\vec{k}\right) \cdot \left(-3\vec{i} - 2\vec{j}\right) = 2 \times (-3) + 0 \times (-2) + 4 \times 0 = -6$

See Exercises 1, 2, and 3.

Angle Between Two Vectors

The angle $\theta, (0 \leq \theta \leq \pi)$, between two vectors can be found using the definition of the dot product:

$$\vec{a} \cdot \vec{b} = \mid \vec{a} \mid\mid \vec{b} \mid \cos\theta.$$

Rearranging,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\mid \vec{a} \mid\mid \vec{b} \mid}$$

and

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\mid \vec{a} \mid\mid \vec{b} \mid}\right).$$

Examples

1. If $\vec{a} = (2,3,1)$ and $\vec{b} = (5,-2,2)$ find the angle θ, between $\vec{a}$ and $\vec{b}$

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\,\vec{a}\,||\,\vec{b}\,|}\right)$$
$$\vec{a}\cdot\vec{b} = (2,3,1)\cdot(5,-2,2) = 6$$
$$|\,\vec{a}\,| = \sqrt{2^2+3^2+1^2} = \sqrt{14}, |\,\vec{b}\,| = \sqrt{25+4+4} = \sqrt{33}$$
$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{33}\times\sqrt{14}}\right)$$
$$= \cos^{-1}(0.2791)$$
$$\theta = 73.8^\circ.$$

The angle between $\vec{a}$ and $\vec{b}$ is 73.8°.

2. Find the angle θ, between $\vec{a}\,(1,0,1)$ and $\vec{b}\,(-2,-1,1)$.

$$\vec{a}\cdot\vec{b} = (1,0,1)\cdot(-2,-1,1) = -1$$
$$|\,\vec{a}\,| = \sqrt{2}$$
$$|\,\vec{b}\,| = \sqrt{6}$$
$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\,\vec{a}\,||\,\vec{b}\,|}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}\times\sqrt{6}}\right) = \cos^{-1}(-0.2887)$$
$$\theta = 106.8^\circ.$$

The angle between $\vec{a}$ and $\vec{b}$ is 106.8°.
See Exercises 4 and 5.

Exercise 1

Calculate the dot product of:
(a) $(2,5,-1)$ and $(4,1,1)$
(b) $3\vec{i}$ and $5\vec{j}$
(c) $5\vec{k}$ and $\left(\vec{j}+2\vec{k}\right)$
Answers:
(a) 12 (b) 0 (c) 10

Exercise 2

Find:
(a) $(2,0,4)\cdot(-3,1,3)$
(b) $(0,5,1)\cdot(4,0,0)$
(c) $\left(2\vec{i}+3\vec{k}\right)\cdot\left(7\vec{i}+2\vec{j}+4\vec{k}\right)$
Answers:
(a) 6 (b) 0 (c) 26

Exercise 3

Which of the following vectors are perpendicular?

(a) $(5, 2, 3)$
(b) $(0, 1, -1)$
(c) $(-2, 2, 2)$
Answers:
(a) and (c), (b) and (c)

Exercise 4

Find the angle between the following pairs of vectors:

(a) $(1, 2, 3)$ and $(4, -1, 0)$
(b) $(2, 1, -2)$ and $(1, 5, -1)$
(c) $(0, 5, 1)$ and $(2, 0, 0)$
(d) $(1, -2, 3)$ and $(-4, 1, -3)$
(e) $(2, 1, -2)$ and $(0, 4, 0)$
(f) $(0, 3, 0)$ and $(0, 1, 0)$
Answers:
(a) $82.6°$ (b) $54.7°$ (c) $90°$ (d) $141.8°$ (e) $70.5°$ (f) $0°$

Exercise 5

If $\vec{a} = (2, 2, 2)$, $\vec{b} = (3, 2, -1)$, and $\vec{c} = (-1, 4, 1)$,

(a) Show $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$
(b) Rearranging $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ gives $\vec{a} \cdot \left(\vec{b} - \vec{c}\right) = 0$. As $\vec{b} \neq \vec{c}$ what is the relationship between $\vec{a}$ and $\left(\vec{b} - \vec{c}\right)$?
Answers:
(a) $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 8 \quad (b)$ $\vec{a}$ is perpendicular to $\left(\vec{b} - \vec{c}\right)$.