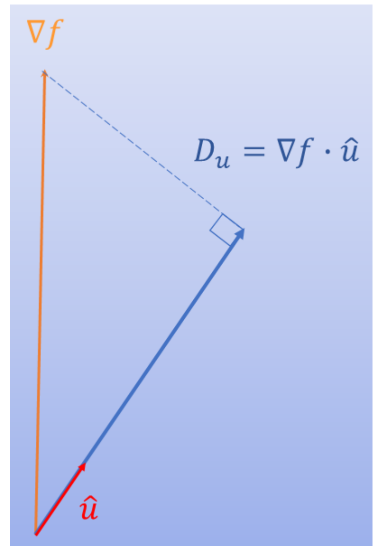
V11: Directional Derivatives



You will recall that the derivative of a function gives us the gradient or rate of change of the function. The rate of change of a function such as $z = f(x, y)$ can be found by partial differentiation; $\frac{\partial f}{\partial x}$ gives the rate of change of the function f, with respect to x, that is, the gradient of the graph as we move in the x direction. Keep in mind that the graph of $z = f(x, y)$ is a surface in three dimensional space and $\frac{\partial f}{\partial y}$ gives the rate of change of f with respect to y, that is, the gradient of the graph as we move in the y direction.

Directional Derivatives

We can find the rate of change of a function in any direction (not just in the direction of the $x-$ axis or $y-$ axis) by finding the directional derivative.

The directional derivative of a function f in the direction of a vector $\vec{u}$ is denoted by D_u and is given by:[1]

[1] Note that the function f may be of 2 or more variables.

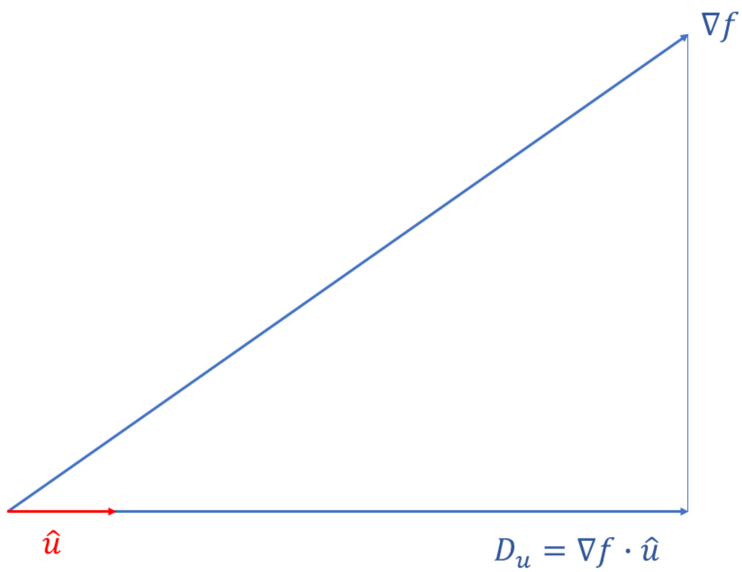
$$D_u = \nabla f \cdot \hat{u}$$

where $\hat{u}$ is a unit vector[2] in the direction of the vector $\vec{u}$. The following figure illustrates the meaning of the directional derivative:

[2] A unit vector is a vector of magnitude 1. For a vector $\vec{u}$ a unit vector in the direction of $\vec{u}$ is given by

$$\hat{u} = \frac{1}{|\vec{u}|}\vec{u}$$

where $|\vec{u}|$ is the magnitude of the vector $\vec{u}$.



Example 1

Suppose that we wish to find the directional derivative of $f(x, y) = x^2 + y^2$ at the point $(2, 3)$ in the direction of the vector $\vec{u} = 4\vec{i} - 3\vec{j}$.

Solution:

The first step is to find the vector "grad f", symbolised thus ∇f.

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}.$$

In this particular case

$$\begin{aligned}\nabla f &= 2x\vec{i} + 2y\vec{j} \\ &= 4\vec{i} + 6\vec{j}\end{aligned}$$

at the point where $x = 2$ and $y = 3$.

The next step is to find the dot product of this vector ∇f, and $\hat{u}$, the unit vector in the direction of $\vec{u}$.

In this case $\vec{u} = 4i - 3j$, therefore $\hat{u} = \frac{1}{5}(4i - 3j)$

Hence the directional derivative of f in the direction of u is

$$\begin{aligned}D_u &= (4i + 6j) \cdot \left(\frac{4}{5}i - \frac{3}{5}j\right) \\ &= \frac{16}{5} - \frac{18}{5} \\ &= -\frac{2}{5}.\end{aligned}$$

To generalise the above, the directional derivative of a function, f, in the direction of u is

$$D_u = \nabla f \cdot \hat{u}$$

where

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.$$

Since the directional derivative relies on a dot product, (remember $\vec{a}.\vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta$), it will be a maximum when $\cos\theta$ is maximum (that is, $\cos\theta = 1$), so the directional derivative will be maximum when $\theta = 0$.

In other words, we can say that

- D_u will be maximum when $\vec{u}$ and ∇f are in the same direction.
- D_u will have its greatest negative value when u and ∇f are in opposite directions (when $\theta = \pi$).
- D_u will be zero when u and ∇f are at right angles.

Example 2

If $f(x, y, z) = x^2 + y^2 + xyz$, find a unit vector $\hat{u}$ such that the rate of change of f at $(2, 3, -1)$ in the direction of u is maximum.

Solution:

$$\begin{aligned}\nabla f =& \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \\ =& (2x + yz)\,\hat{i} + (2y + xz)\,\hat{j} + (xy)\,\hat{k} \\ =& \hat{i} + 4\hat{j} + 6\hat{k}\end{aligned}$$

For the directional derivative to be maximum, $u = \nabla f$
Therefore $u = \hat{i} + 4\hat{j} + 6\hat{k}$ and $\hat{u} = \frac{1}{\sqrt{53}}\left(\hat{i} + 4\hat{j} + 6\hat{k}\right)$.

Exercise

1. Find the directional derivative of the given function at the given point in the direction of the indicated vector:
a) $f(x, y) = xy^2$, $(3, 2)$, $4\hat{i} + 3\hat{j}$
b) $f(x, y) = e^{xy}$, $(0, 2)$, $\hat{i}$
c) $f(x, y, z) = x^2y^3z$, $(2, -1, 3)$, $\hat{i} - 2\hat{j} - 2\hat{k}$
Answers:
a) $\frac{52}{5}$ b) 2 c) $-\frac{76}{3}$.

2. Find the unit vector in the direction in which f increases most rapidly at $P(1, \pi/2)$ for $f(x, y) = x^2 + \cos xy$.
Answer:
$\hat{u} = 0.394\hat{i} - 0.919\hat{j}$.