T7: Trigonometric Equations

Degrees	Radians
0	0
30	$\frac{\pi}{6}$
45	$\frac{\pi}{4}$
60	$\frac{\pi}{3}$
90	$\frac{\pi}{2}$
180	π
270	$\frac{3\pi}{2}$
360	2π

This module describes how to solve trigonometric equations over a prescribed domain. An example of such an equation is: Solve

$$\cos\theta = 0.5$$

if $-\pi \leq \theta \leq 3\pi$.

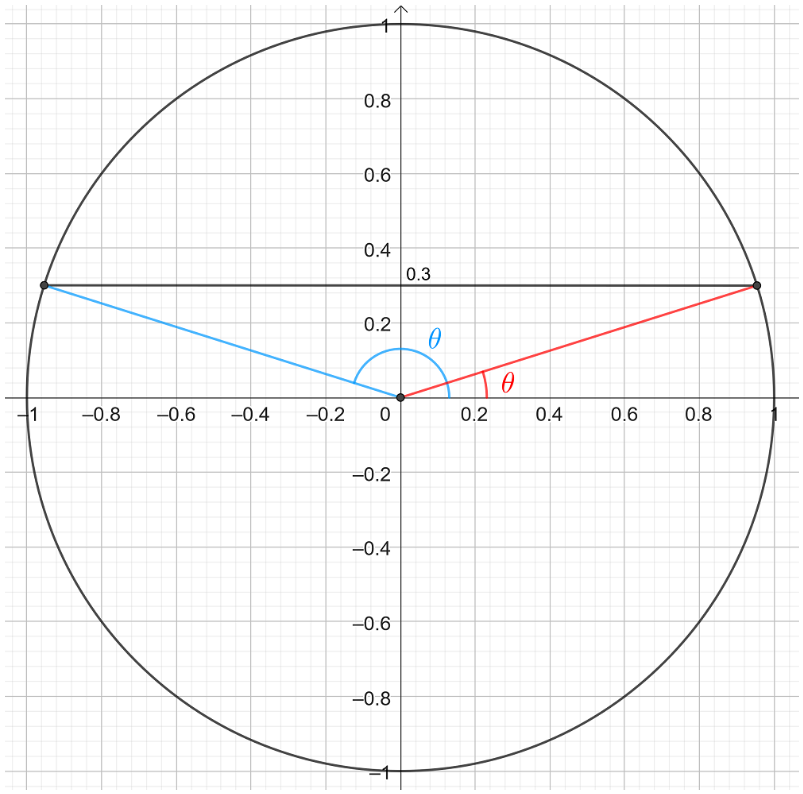
Solving these types of problem is easier if you can identify the values of trigonometric functions on a unit circle as shown in the example below. You also need to understand angles in degrees and radians.

Example 1

Given that $\sin\theta = 0.3$, find all values of θ in the domain $\{\theta : 0^\circ \leq \theta \leq 360^\circ\}$.

Solution:

Draw a picture showing the possible positive values for θ as shown below:



There are at least two solutions shown in red and blue. We have θ_r (in red) which is given by

$$\begin{aligned} \theta_r &= \sin^{-1}(0.3) \\ &= 17.46^\circ. \text{ (from calculator)} \end{aligned}$$

The other angle θ_b (in blue) may be found by symmetry:

$$\begin{aligned} \theta_b &= 180^\circ - \theta_r \\ &= 180^\circ - 17.46^\circ \\ &= 162.54^\circ. \end{aligned}$$

Hence the solutions are $\theta = 17.46^\circ, 162.54^\circ$.

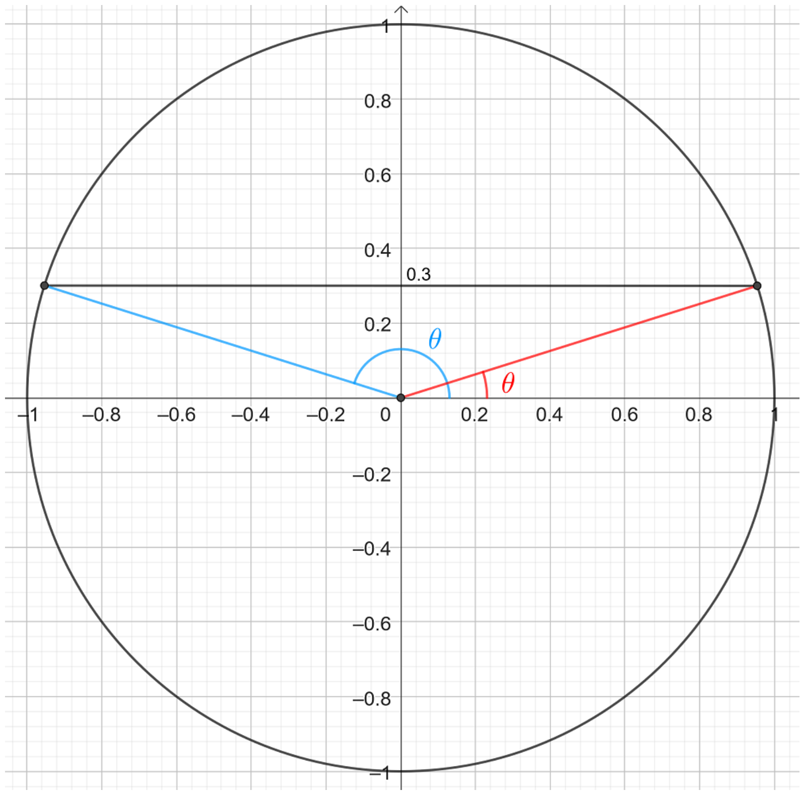
Example 2

Given that $\sin\theta = 0.3$, find all values of θ such that $0^\circ \leq \theta \leq 500^\circ$.

Solution:

Note that this is the same problem as in Example 1 except that the domain in which we are looking for solutions has been extended to 500°.

Draw a picture showing the possible positive values for θ as shown below:



From Example 1 we know $\theta = 17.46°$or $162.54°$. If we add $360°$ to each of these, they will still satisfy $\sin\theta = 0.3$. The question is, are they in the domain of interest?

Adding $360°$ to the solution $\theta = 17.46°$ found in Example 1 gives

$$\begin{aligned}\theta &= 17.46° + 360° \\ &= 377.46°\end{aligned}$$

which is less than $500°$ and so a solution to Example 2. Adding $360°$ to the other solution to Example 1 gives

$$\begin{aligned}\theta &= 162.54° + 360° \\ &= 522.54°\end{aligned}$$

which is bigger than $500°$ and so not a solution to Example 2.

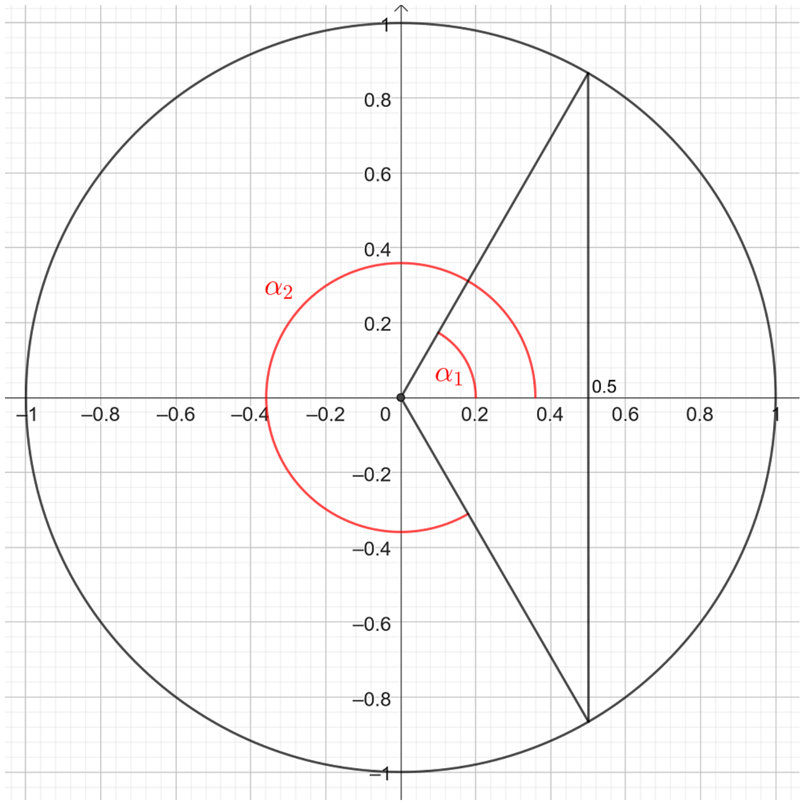
Hence the solutions are $\theta = 17.46°$, $162.54°$ and $377.46°$.

Example 3

Solve $\cos\alpha = 0.5$ over the domain $\{\alpha : 0 \leq \alpha \leq 2\pi\}$.

Solution:

Draw a picture showing the possible positive values for α as shown below:



Referring to the figure,

$$\begin{aligned} \alpha_1 &= \cos^{-1}(0.5) \\ &= \frac{\pi}{3} \text{ (from calculator)} \end{aligned}$$

and from symmetry of the unit circle,

$$\begin{aligned} \alpha_2 &= 2\pi - \alpha_1 \\ &= 2\pi - \frac{\pi}{3} \\ &= \frac{5\pi}{3}. \end{aligned}$$

Hence the solution is $\alpha = \pi/3$ and $5\pi/3$.

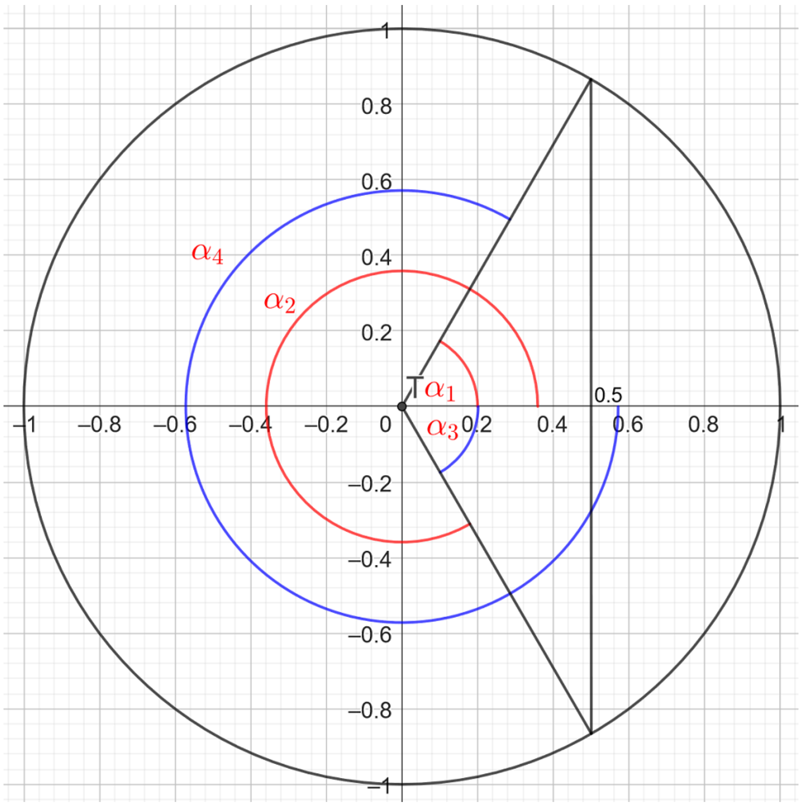
Example 4

Solve $\cos\alpha = 0.5$ over the domain $\{\alpha : -2\pi \leq \alpha \leq 2\pi\}$.

Solution:

Note this is the same problem as in Example 3 but the domain has been extended.

Draw a picture showing the possible positive and negative values for α as shown below:



The two positive angles (shown in red) are the same as those in Example 3, namely

$$\alpha_1 = \frac{\pi}{3}$$
$$\alpha_2 = \frac{5\pi}{3}.$$

The negative angles (shown in blue) are α_3 and α_4. Using the results from Example 3 and symmetry we have

$$\begin{aligned}\alpha_3 &= -\alpha_1 \\ &= -\frac{\pi}{3}\end{aligned}$$

and

$$\begin{aligned}\alpha_4 &= -(2\pi - \alpha_1) \\ &= \alpha_1 - 2\pi \\ &= \frac{\pi}{3} - 2\pi \\ &= -\frac{5\pi}{3}.\end{aligned}$$

Hence the solution is $\alpha = -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$.

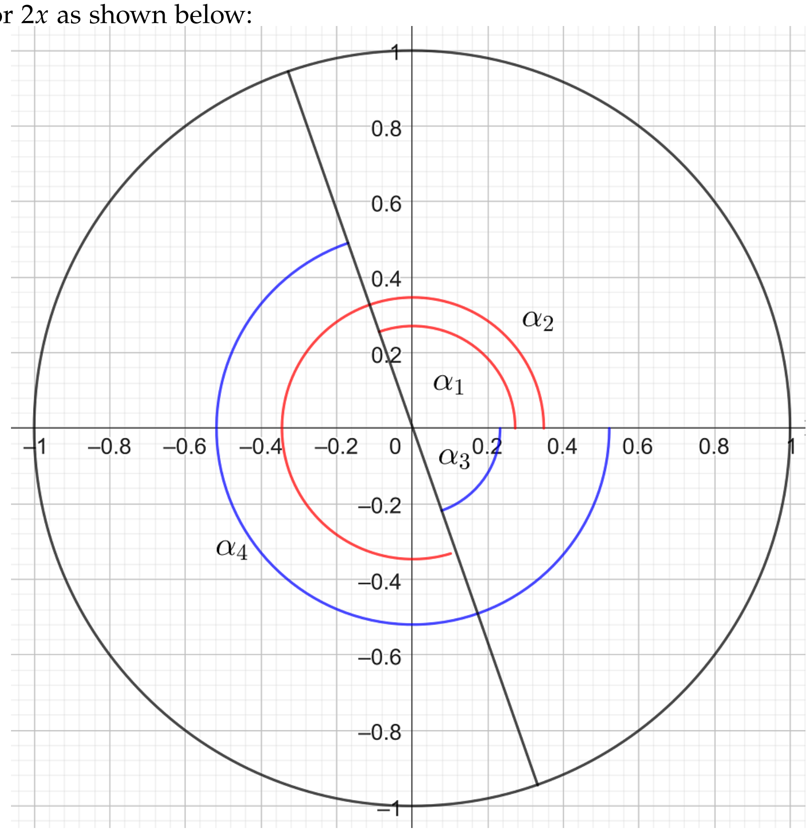
Example 5

Solve $\tan(2x) = -3$ over the domain $\{x : -90^\circ \leq x \leq 180^\circ\}$.

Solution:

Since the variable is $2x$, we have to redefine the domain to $\{2x : -180^\circ \leq 2x \leq 360^\circ\}$.

Draw a picture showing the possible positive and negative values for $2x$ as shown below:



There are two positive angles α_1, α_2 (shown in red) and two negative angles α_3, α_4 (shown in blue) that may lie within $-180^\circ \leq 2x \leq 360^\circ$.

We have

$$\begin{aligned} \tan(2x) &= -3 \\ 2x &= \tan^{-1}(-3) \\ &= -71.57^\circ \text{ (from calculator).} \end{aligned}$$

Referring to the figure, we see that

$$\alpha_3 = -71.57^\circ$$

and

$$\begin{aligned} \alpha_4 &= \alpha_3 - 180^\circ \\ &= -71.57^\circ - 180^\circ \\ &= -251.57^\circ. \end{aligned}$$

Note that this is outside of the domain for $2x$ and so is rejected. That is we discard α_4.

Using the symmetry of the unit circle, we have

$$\begin{aligned} \alpha_1 &= 360^\circ + \alpha_4 \\ &= 360^\circ - 251.57^\circ \\ &= 108.43^\circ \end{aligned}$$

and using the figure,

$$\begin{aligned} \alpha_2 &= \alpha_1 + 180^\circ \\ &= 108.43^\circ + 180^\circ \\ &= 288.43^\circ. \end{aligned}$$

Hence

$$\begin{aligned} 2x &= \alpha_3,\ \alpha_1,\ \alpha_2 \\ &= -71.57^\circ,\ 108.43^\circ,\ 288.43^\circ \end{aligned}$$

and the solution is

$$x = -35.79^\circ,\ 54.22^\circ,\ 144.22^\circ.$$

Exercises

Solve the following equations:

a) If $\sin\phi = 0.25$ find ϕ for $0^\circ \leq \phi \leq 180^\circ$.
b) If $\tan\phi = 0.8$ find ϕ for $0^\circ \leq \phi \leq 360^\circ$.
c) If $\cos\phi = 0.4$ find ϕ for $0^\circ \leq \phi \leq 360^\circ$.
d) If $\cos\phi = -0.4$ find ϕ for $-180^\circ \leq \phi \leq 360^\circ$.
e) If $\tan\phi = -1.5$ find ϕ for $-180^\circ \leq \phi \leq 360^\circ$.
f) If $\cos\phi = -0.3$ find ϕ for $0^\circ \leq \phi \leq 360^\circ$.

Answers

All answers are in degrees.

a) 14.5°, 165.5° b) 38.7°, 218.7° c) 66.4°, 293.6°
d) -113.6°, 113.6°, 246.4° e) -56.3°, 123.7°, 303.7° f) 107.5°, 252.5°.