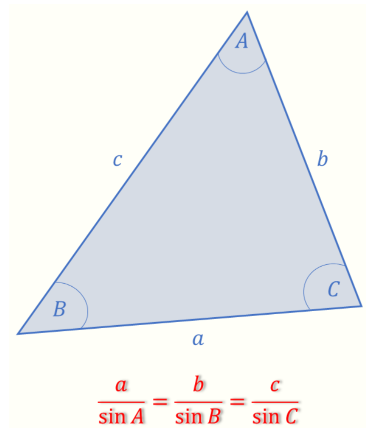
T3: The Sine Rule



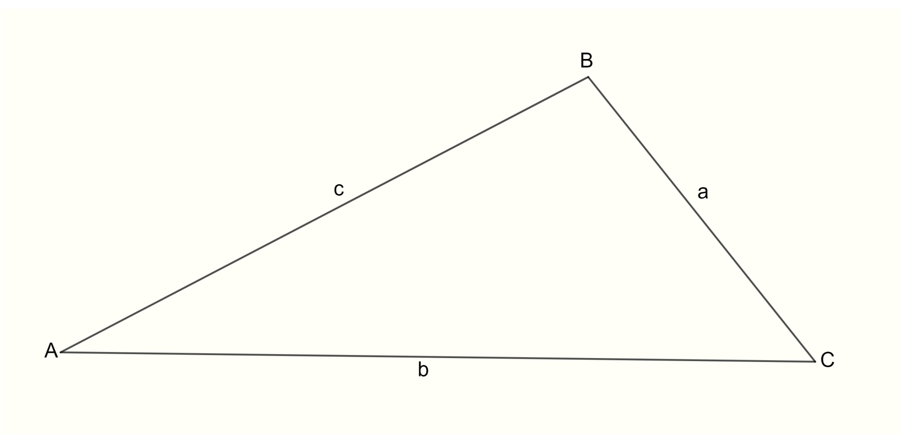
The Sine rule can be used to find angles and sides in any triangle (not just a right-angled triangle) when given:

1. One side and any two angles or
2. Two sides and an angle opposite one of the given sides.

The Sine Rule

In the triangle ABC below:

- angles $A, B, C,$ are the angles at the vertices A, B, C respectively
- a, b, c are the side lengths opposite the angles A, B, C respectively.



The sine rule states:

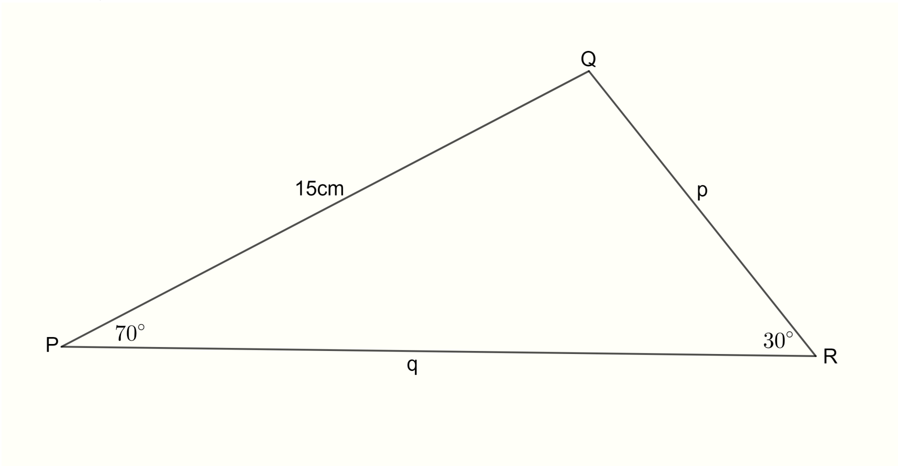
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Examples

1. In triangle PQR find:



a. the side length p
b. the side length q.
Solution a.
Use the sine rule in the form

$$\begin{aligned} \frac{p}{\sin P} &= \frac{r}{\sin R} \\ \frac{p}{\sin 70^\circ} &= \frac{15}{\sin 30^\circ} \\ p &= \frac{15 \times \sin 70^\circ}{\sin 30^\circ} \\ p &= 28.2\,. \end{aligned}$$

The side length is $p = 28.2\,cm$.
Solution b)
Angle Q is found using the fact that the sum of the three interior angles of a triangle add to 180°:

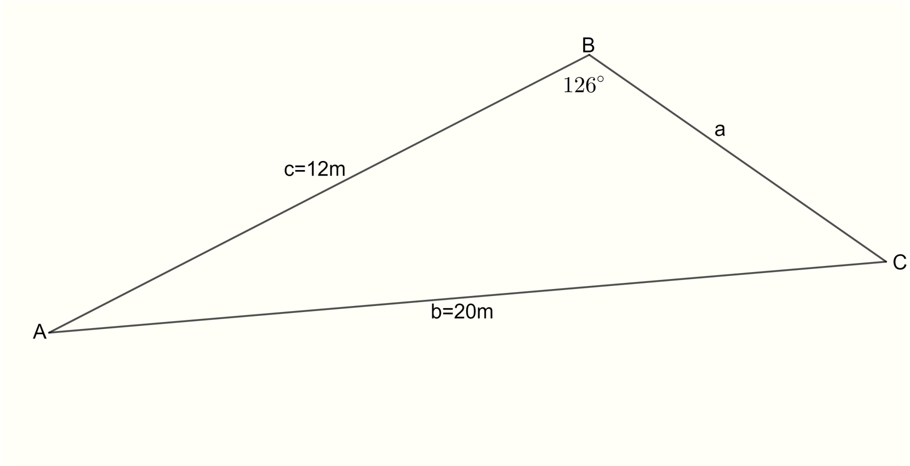
$$\begin{aligned} \angle Q &= 180^\circ - (70^\circ + 30^\circ) \\ &= 80^\circ. \end{aligned}$$

From the Sine rule

$$\begin{aligned} \frac{q}{\sin Q} &= \frac{r}{\sin R} \\ \frac{q}{\sin 80^\circ} &= \frac{15}{\sin 30^\circ} \\ q &= \frac{15 \times \sin 80^\circ}{\sin 30^\circ} \\ q &= 29.6\,. \end{aligned}$$

The length of the side $q = 29.6\,cm$.

2. In triangle ABC find:



a. Angle C
b. Angle A
c. Side length a.
Solution a.
Use the sine rule in the form

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The relevant part of the formula is

$$\begin{aligned}\frac{\sin B}{b} &= \frac{\sin C}{c}\\ \frac{\sin 126^\circ}{20} &= \frac{\sin C}{12}\\ \frac{\sin 126^\circ}{20} \times 12 &= \sin C\\ \sin C &= 0.485\\ C &= \sin^{-1} 0.485\\ C &= 29^\circ.\end{aligned}$$

Solution b.

$$\begin{aligned}\angle A &= 180^\circ - (126^\circ + 29^\circ)\\ &= 25^\circ.\end{aligned}$$

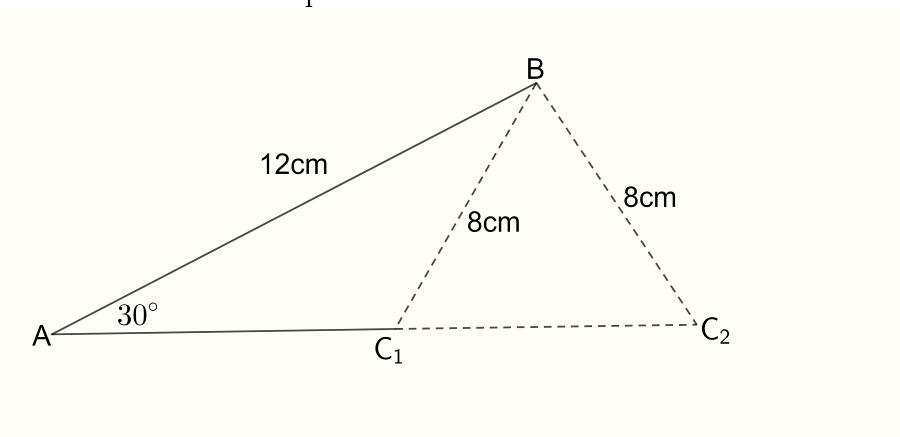
Solution c.
Use the sine rule in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The relevant part of the formula is

$$\begin{aligned} \frac{a}{\sin A} &= \frac{b}{\sin B} \\ \frac{a}{\sin 25^\circ} &= \frac{20}{\sin 126^\circ} \\ a &= \frac{20}{\sin 126^\circ} \times \sin 25^\circ \\ a &= 10.4m \end{aligned}$$

3. Given a a triangle ABC with angle $A = 30^\circ$, side $c = 12$ cm and side $a = 8$ cm, find angle C.
Solution:
In this case there are two possible solutions as shown below.



This is called the ambiguous case of the sine rule.
To solve the problem, use the sine rule in the form:

$$\begin{aligned} \frac{\sin A}{a} &= \frac{\sin C}{c} \\ \frac{\sin 30^\circ}{8} &= \frac{\sin C}{12} \\ \frac{\sin 30^\circ}{8} \times 12 &= \sin C \\ \sin C &= \frac{1}{16} \times 12 \\ &= 0.75 \\ C &= \sin^{-1}(0.75) \\ &= 48.6^\circ \text{or } 131.4^\circ. \end{aligned}$$

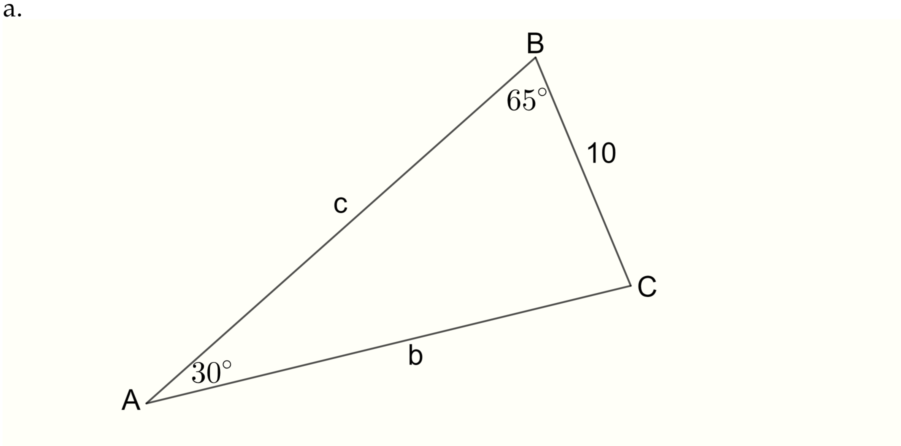
If $C = 48.6^\circ$, this solution gives the triangle ABC_2.
If $C = 131.4^\circ = (180^\circ - 48.6^\circ)$, this solution gives the triangle ABC_1.

Note: In any non-right-angled triangle, where two sides and the non-included angle are given, check for the ambiguous case.
If the angle is acute and the length of the side adjacent to the angle is greater than the length of the side opposite the angle there may be 2 possible solutions.

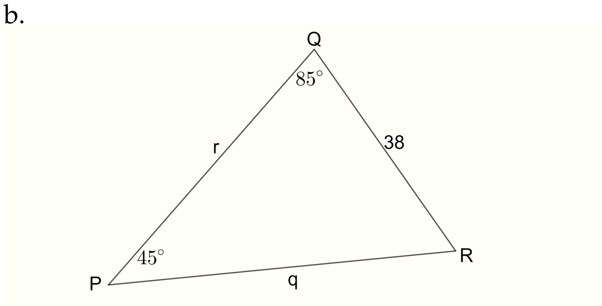
Exercise 1

For the following triangles find the unknown sides.

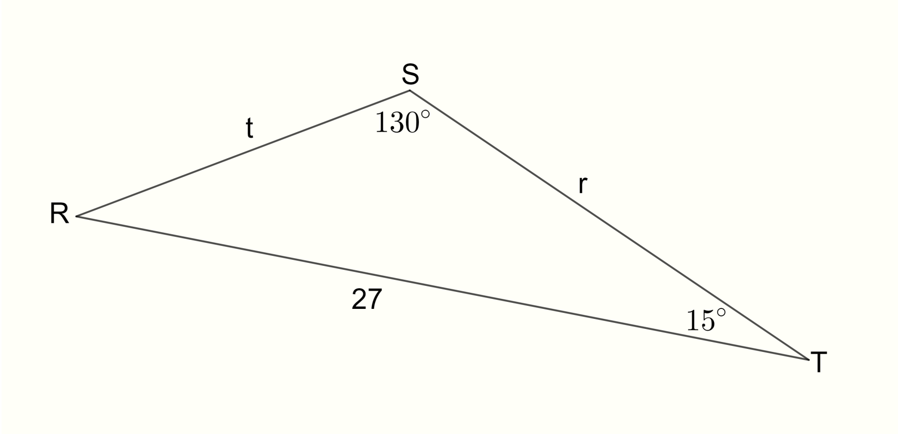
a.



b.



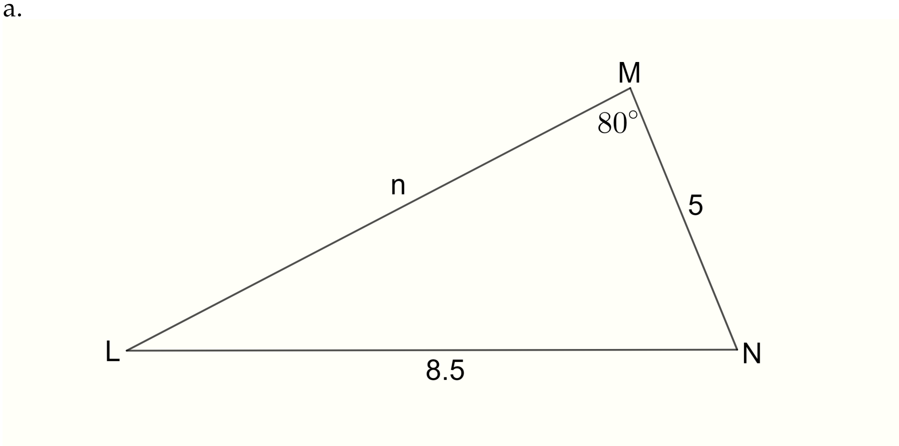
c.



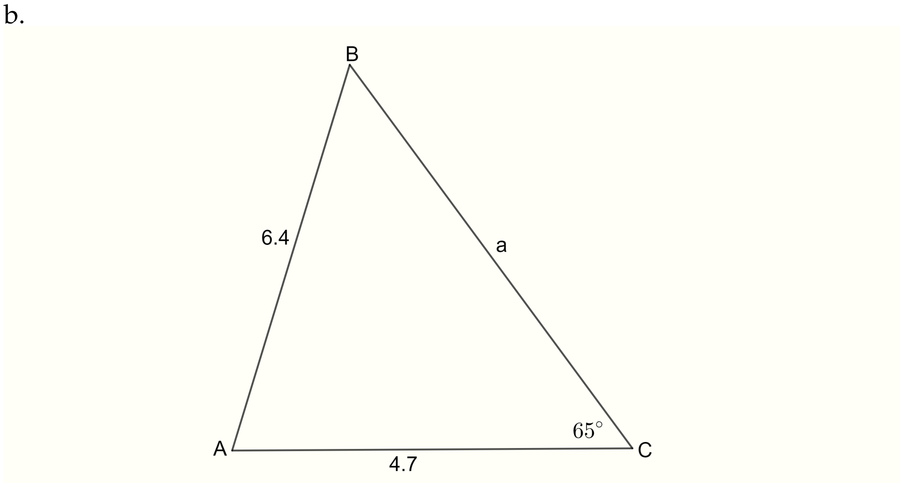
Exercise 2

For the following triangles find all unknown angles and sides.

a.



b.



c. Given triangle ABC, where angle $A = 35$, side $a = 16$ and side $c = 21$, find the magnitude of angles B and C and the length of side b.

Answers

Exercise 1
a. $b = 18.1$, $c = 19.9$
b. $q = 53.5$, $r = 41.2$
c. $t = 9.1$, $r = 20.2$
Exercise 2
a. $L = 35.4°$, $N = 64.6°$, $n = 7.8$
b. $B = 41.7°$, $A = 73.3°$, $a = 6.8$
c. Ambiguous case: $C = 131.2°$, $B = 13.8°$, $b = 6.7$ or $C = 48.8$, $B = 96.2$, $b = 27.7$