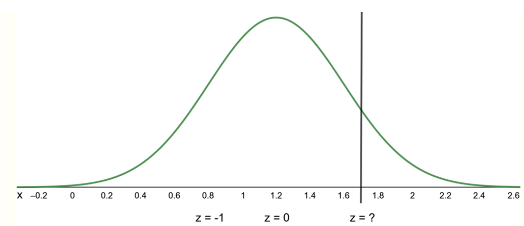
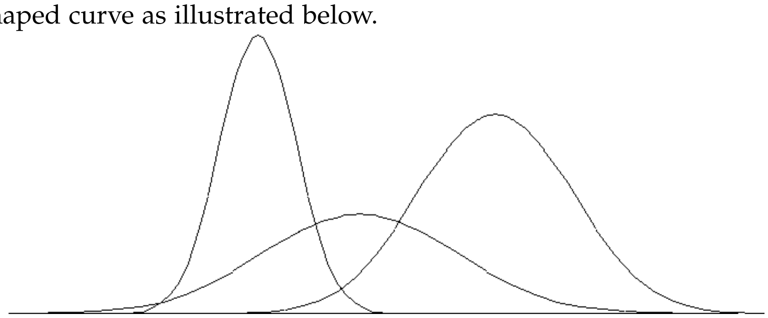
S11. *Probability and the Normal Distribution*



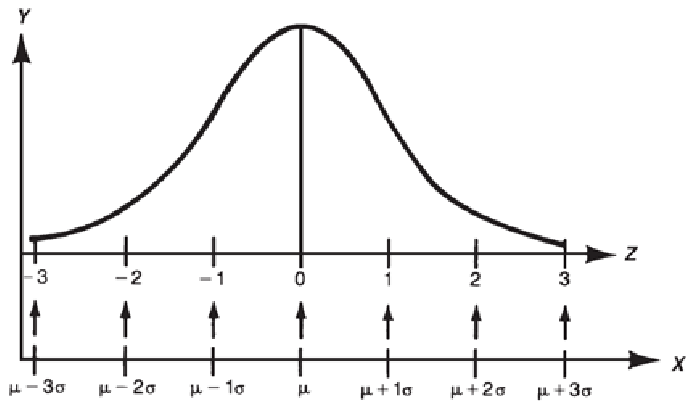
$$z = \frac{x - \mu}{\sigma}$$

$$Pr(z) < 1.7$$

Even when data follows a normal distribution, different data sets will have their own mean and standard deviation and a different bell shaped curve as illustrated below.



But every score in a normally distributed data set, regardless of the shape, has an equivalent score in the standard distribution. The mean of a normal distribution corresponds to a standardised score of 0 and we can see that $\mu \pm \sigma \rightarrow \pm 1$, $\mu \pm 2\sigma \rightarrow \pm 2$ and $\mu \pm 3\sigma \rightarrow \pm 3$.



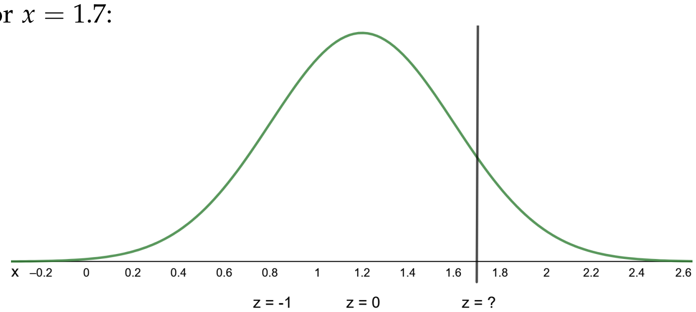
z−Scores

For other values of the mean and standard deviation, we can use the formula

$$z = \frac{x - \mu}{\sigma}$$

to find z scores (where the z score is the number of standard deviations away from the mean).

For example, if we have a Normal Distribution with a mean of 1.2 and a standard deviation of 0.4, then to find the standardised score for $x = 1.7$:



$$\begin{aligned} z &= \frac{(x - \mu)}{\sigma} \\ &= \frac{(1.7 - 1.2)}{0.4} \\ &= 1.25 \end{aligned}$$

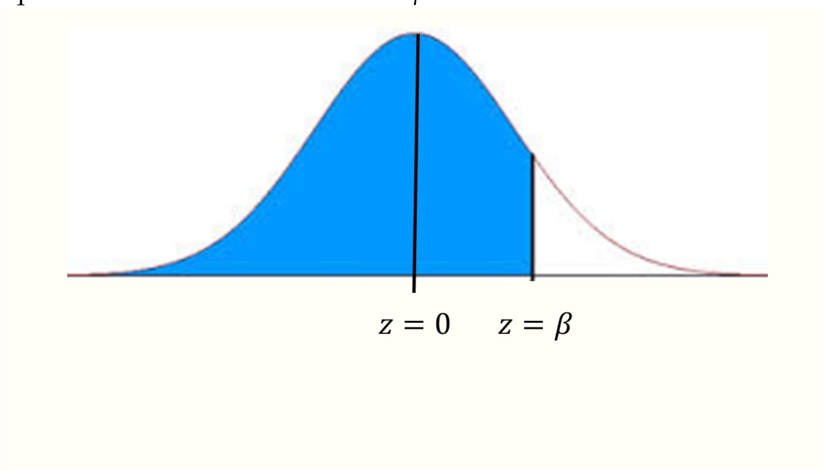
A score of 1.7 in the distribution with mean 1.2 and standard deviation 0.4 is equivalent to a standardised score of 1.25. Alternatively, we could say that the score 1.7 is 1.25 standard deviations above the mean for that distribution.

Calculating Probabilities Using z−Scores

Once we have converted the scores of our distribution into standard scores or z-scores we can use normal distribution tables to calculate precise percentages and probabilities.

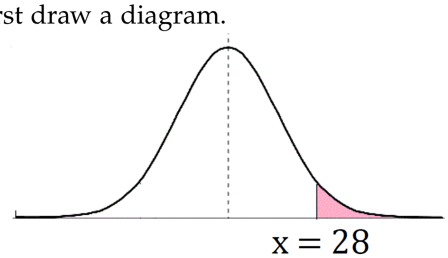
The normal distribution is a continuous distribution, so we can find the probability that x is greater than or less than a particular value, but not that x is equal to a particular value.

Because the total area under the standardised curve is 1, $\Pr(z < \beta)$ is equivalent to the area to the left of β.



Example 1

If the mean maximum temperature for Melbourne in January is 25.9°C with a standard deviation of 2.1° what is the probability that the mean maximum temperature for January 2015 will be above 28°C?
First draw a diagram.

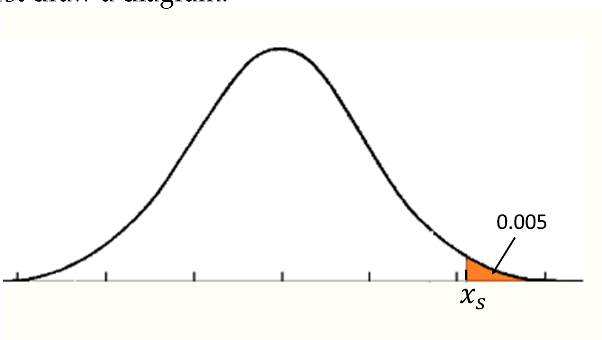


$$
\begin{aligned}
x &= 28 \\
z &= \frac{(x-\mu)}{\sigma} \\
&= \frac{(28-25.9)}{2.1} \\
&= 1. \\
\Pr(x > 28) &= \Pr(z > 1) \\
&= 1 - \Pr(z < 1) \\
&= 1 - 0.8413 \quad \text{(0.8413 is from the table at end of this module)} \\
&= 0.1587.
\end{aligned}
$$

That is, the probability that the mean maximum temperature for January 2015 will be above 28°C is 0.1587.

Example 2

The top 0.5% of students applying for Stato university are given full scholarships. If the mean score on the entrance exam is 372 and the standard deviation is 40, what mark is needed to obtain a scholarship?
First draw a diagram.



The area to the right of $x_s = 0.005$.[1]
We know that $\Pr(x > x_s) = 0.005$ but we must first find the z- score corresponding to this. Let's call it z_s. Then

$$\Pr(z > z_s) = 0.005$$

and

$$
\begin{aligned}
\Pr(z < z_s) &= 1 - 0.005 \\
&= 0.9950.
\end{aligned}
$$

Using the table at the end of this module, we look up 0.995 in the body of the table and see that the corresponding z- score is $z_s = 2.57$ or $z_s = 2.58$. We will take $z_s = 2.58$.[2] Now using the formula we have

[1] We want to find x_s.

[2] You could take the average of 2.57 and 2.58 in this case, that is 2.575 but two decimal places of accuracy is usually enough.

$$\begin{aligned} z_s &= \frac{x_s - \mu}{\sigma} \\ 2.58 &= \frac{x_s - 372}{40}. \\ \text{Rearranging we have, } x_s &= 2.58 \times 40 + 372 \\ &= 475.2\,. \end{aligned}$$

So applicants who score more than 475.2 will get a scholarship.

Exercises

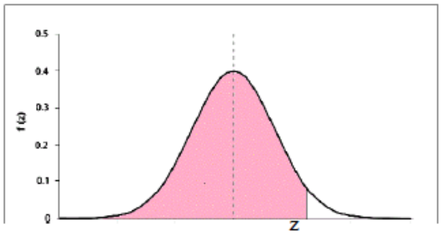
1. If a population has a mean I.Q. of 100 and a standard deviation of 15, find:
(a) the probability that an individual chosen at random will have an I.Q. between 110 and 130.
(b) the probability that an individual chosen at random will have an I.Q. greater than 87.
Answer: (a) 0.2297 (b) 0.8069

2. A coffee machine is regulated to deliver 200 mL per cup. In fact, the amount of coffee varies, following a normal distribution with a mean of 200 mL and a standard deviation of 10 mL.
(a) What is the probability that a cup contain less than 195 mL?
(b) What is the probability that a cup will contain more than 220 mL?
(c) What is the probability of a cup containing between 195 and 215 mL?
Answer: (a) 0.3085 (b) 0.0228 (c) 0.6247

3. The heights of a group of men follow a normal distribution with a mean of 180 cm and a standard deviation of 6 cm.
(a) What is the probability that a man chosen from this group is less than 185 cm tall?
(b) If the tallest 10% of this group are automatically eligible for a basketball team, what is the qualifying height?
Answer: 0.7977 (b) 187.69 cm

Table for Calculating z-Values

The table below gives the probability (the shaded area to the left) for a particular z-value.



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9031	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9958	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986