RMIT UNIVERSITY

WORKED SOLUTIONS

NM2.6 NUMERICAL METHODS: RUNGE-KUTTA METHOD

Question

For the second order differential equation

$$\frac{d^2y}{dx^2} + y = x \qquad y(0) = 1 \qquad y'(0) = 0$$

Apply two steps of the second order Runge-Kutta scheme to obtain the approximate solution to y at $x = 1.0$.

$$\frac{dy}{dx} = f(x, y, u) \qquad \frac{du}{dx} = g(x, y, u)$$

$$k_1 = hf(x_n, y_n, u_n) \qquad l_1 = hg(x_n, y_n, u_n)$$

$$k_2 = hf(x_n + h, y_n + k_1, u_n + l_1) \qquad l_2 = hg(x_n + h, y_n + k_1, u_n + l_1)$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \qquad u_{n+1} = u_n + \frac{1}{2}(l_1 + l_2)$$

Worked Solution

Write this 2nd order DE as a 1st order DE

Let $u = \frac{dy}{dx}$, then $\frac{du}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

Substituting into $\frac{d^2y}{dx^2} + y = x$: $\frac{du}{dx} + y = x \Rightarrow \frac{du}{dx} = x - y$

Note: dy/dx is a function of x and u, so

$$\frac{dy}{dx} = f(x, y, u) = u$$

du/dx is a function of x and y, so

$$\frac{du}{dx} = g(x, y, u) = x - y$$

2 steps ($n=0, n=1$) required for solution at $x=0.5$
Initial condition is $x=0$, hence $h = \frac{0.5}{2} = 0.25$

$n=0$: $x_0 = 0, y_0 = 1, u_0 = 0, h = 0.25$

$K_1 = hf(x_0, y_0, u_0) = hu_0$
$\Rightarrow K_1 = 0.25 \times 0 = 0$

$K_2 = hf(x_0+h, y_0+K_1, u_0+l_1)$
$K_2 = 0.25(0 - 0.25) = -0.0625$

$l_1 = hg(x_0, y_0, u_0) = 0.25(0-1)$
$\Rightarrow l_1 = -0.25$

$l_2 = hg(x_0+h, y_0+K_1, u_0+l_1)$
$l_2 = 0.25(0.25 - 1) = -0.1875$

$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$

$y_1 = 1 + 0.5(0 + (-0.0625))$
$y_1 = 0.96875$

$u_1 = u_0 + \frac{1}{2}(l_1 + l_2)$

$u_1 = 0 + 0.5(-0.25 - 0.1875)$
$u_1 = -0.21875$

$n=1$: $x_1 = 0.25, y_1 = 0.96875, u_1 = -0.21875, h = 0.25$

$K_1 = hf(x_1, y_1, u_1) = hu_1$
$K_1 = 0.25 \times -0.21875 = -0.0546875$

$l_1 = hg(x_1, y_1, u_1)$
$l_1 = 0.25(0.25 - 0.96875)$
$l_1 = -0.1796875$

$K_2 = hf(x_1+h, y_1+K_1, u_1+l_1)$
$K_2 = 0.25(-0.21875 + -0.1796875)$
$K_2 = -0.0996093 75$

$l_2 = hg(x_1+h, y_1+K_1, u_1+l_1)$
$l_2 = 0.25(0.5 - 0.3984375)$
$l_2 = 0.02539062 5$

$y_2 = y_1 + \frac{1}{2}(K_1 + K_2) = 0.96875 + 0.5(-0.0546875 - 0.0996093 75)$

$y_2 = 0.8916015625$ //