RMIT UNIVERSITY

WORKED SOLUTIONS

NM2.4: NUMERICAL METHODS: BISECTION METHOD

Question

Use the bisection method to find the root of $x^2 - 3 = 0$ that lies in the interval $[1, 2]$ to a tolerance of 0.1.

Worked Solution

A tolerance of 0.05 is required so that one decimal place accuracy is achieved.

$a_0 = 1, \quad b_0 = 2, \quad \varepsilon = 0.05, \quad f(x) = x^2 - 3$

$$n \geq \frac{\ln\left(\frac{b_0 - a_0}{\varepsilon}\right)}{\ln 2} - 1 = \frac{\ln\left(\frac{2-1}{0.05}\right)}{\ln 2} = 4.3$$

Therefore, we will use 5 iterations, $n = 0, 1, 2, 3, 4$

$\underline{n=0}$: $a_0 = 1, \ b_0 = 2, \ c_0 = \frac{1}{2}(a_0 + b_0) = 1.5, \ f(x) = x^2 - 3$

$$f(c_0) = f(1.5) = (1.5)^2 - 3 = -0.75$$

$$f(b_0) = f(2) = 2^2 - 3 = 1$$

Since $f(b_0) \times f(c_0) = -0.75 \times 1 = -0.75 < 0$

then there is at least one root in the interval $[1.5, 2]$, ie $[c, b]$

Therefore c_0 becomes a_1 in the next iteration

$n=1:$ $a_1 = 1.5$, $b_1 = 2$, $c_1 = \frac{1}{2}(a_1+b_1) = 1.75$, $f(x) = x^2-3$

$$f(c_1) = f(1.75) = (1.75)^2 - 3 = 0.0625$$
$$f(b_1) = f(2) = 2^2 - 3 = 1$$

Since $f(b_1) \times f(c_1) = 0.0625 > 0$
then there is a root in the interval $[1.5, 1.75]$, ie $[a, c]$
Therefore c_1 becomes b_2 in the next iteration.

$n=2:$ $a_2 = 1.5$, $b_2 = 1.75$, $c_2 = \frac{1}{2}(a_2+b_2) = 1.625$, $f(x) = x^2-3$

$$f(c_2) = f(1.625) = (1.625)^2 - 3 = -0.3594$$
$$f(b_2) = f(1.75) = (1.75)^2 - 3 = 0.0625$$

Since $f(b_2) \times f(c_2) = -0.0225 < 0$
then there must be a root in the interval $[1.625, 1.75]$
Therefore c_2 becomes a_3 in the next iteration

$n=3:$ $a_3 = 1.625$, $b_3 = 1.75$, $c_3 = \frac{1}{2}(a_3+b_3) = 1.6875$, $f(x) = x^2-3$
etc, etc, etc, ...

The bisection of the interval is repeated for 5 iterations ($n=0$ to $n=4$)
The results are shown in the table below.

i	*a*	*b*	*c*	*b* – *c*	*f(b)*	*f(c)*	*f(b)×f(c)*	*Action*
0	1	2	1.5	0.5 (> ε)	1	–0.75	< 0	a = c
1	1.5	2	1.75	0.25 (> ε)	1	1.75	> 0	b = c
2	1.5	1.75	1.625	0.125 (> ε)	0.0625	–0.3594	< 0	a = c
3	1.625	1.75	1.6875	0.0625 (> ε)	0.0625	–0.1523	< 0	a = c
4	1.6875	1.75	1.71875	0.03125 (< ε)				

NOTE: After 5 iterations $b-c = 0.03125$, and since $b - c \leq \varepsilon$, or $0.03125 < 0.05$, the iteration is terminated.

The estimate of the root, c_4, is 1.7 to 1 decimal place.