M8: Inverse of a 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In matrix algebra, we can add, subtract and multiply matrices subject to conditions on the matrix shape (or order). While matrix algebra does not have a division operation, there is multiplication by the inverse matrix. This module discuses the concept of an inverse matrix.

Definition

Let I denote the identity matrix. That is the matrix containing ones on the main diagonal and zeros elsewhere. That is

$$I_1 = [1]\,,\; I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\; I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\; I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

If A is a square matrix and B is another square matrix of the same order such that

$$AB = BA = I$$

then we call B the inverse of A. The inverse of A is denoted by the symbol A^{-1}.[1] Hence

$$AA^{-1} = A^{-1}A = I.$$

[1] Note that

$$A^{-1} \neq \frac{1}{A}$$

as division is not defined in matrix algebra.

Not every square matrix has an inverse. If the determinant of a matrix equals zero, the inverse does not exist and the matrix is called singular. If the determinant is unequal to zero the inverse exists and we call the matrix non-singular or invertible.

Inverse of a 2×2 Matrix

If A is a 2×2 matrix, then A^{-1} is also a 2×2 matrix such that:

$$AA^{-1} = A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There is a simple formula to find the inverse of a 2×2 matrix.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the inverse matrix of A is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Note that $ad - bc$ is the determinant of the matrix A. That is $ad - bc = \det A = |A|$. So we can also write

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $\det A = 0$, we have

$$A^{-1} = \frac{1}{0} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

But $1/0$ is undefined and so the inverse does not exist.

Example 1

Find the inverse of the matrix $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$.

Solution:

First check if A is singular.

$$\begin{aligned} \det A &= 2 \times 4 - 2 \times 3 \\ &= 8 - 6 \\ &= 2 \end{aligned}$$

so A is not singular and the inverse exists. Using the formula above,

$$\begin{aligned} A^{-1} &= \frac{1}{\det A} \begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -3/2 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

Check $AA^{-1} = I$ and $A^{-1}A = I$.

$$\begin{aligned}
AA^{-1} &= \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -3/2 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \times 2 + 3 \times (-1) & 2 \times (-3/2) + 3 \times 1 \\ 2 \times 2 + 4 \times (-1) & 2 \times (-3/2) + 4 \times 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
A^{-1}A &= \begin{bmatrix} 2 & -3/2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 2 \times 2 + (-3/2) \times 2 & 2 \times 3 + (-3/2) \times 4 \\ -1 \times 2 + 1 \times 2 & (-1) \times 3 + 1 \times 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

So $A^{-1} = \begin{bmatrix} 2 & -3/2 \\ -1 & 1 \end{bmatrix}$ is the inverse of A.

Example 2

Find the inverse of the matrix $A = \begin{bmatrix} -1 & -2 \\ 4 & 3 \end{bmatrix}$.

Solution:

The determinant,

$$\begin{aligned}
\det A &= -1 \times 3 - (-2) \times 4 \\
&= -3 - (-8) \\
&= 5
\end{aligned}$$

so the matrix A has an inverse. Using the formula,

$$\begin{aligned}
A^{-1} &= \frac{1}{\det A} \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix}.
\end{aligned}$$

Check $AA^{-1} = I$ and $A^{-1}A = I$.

$$\begin{aligned}
AA^{-1} &= \begin{bmatrix} -1 & -2 \\ 4 & 3 \end{bmatrix} \left(\frac{1}{5} \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix} \right) \\
&= \frac{1}{5} \begin{bmatrix} -1 & -2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} -3+8 & -2+2 \\ 12-12 & 8-3 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
A^{-1}A &= \frac{1}{5} \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 4 & 3 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} -3+8 & -6+6 \\ 4-4 & 8-3 \end{bmatrix} \\
&= \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

So $A^{-1} = \frac{1}{5} \begin{bmatrix} 3 & 2 \\ -4 & -1 \end{bmatrix}$.

Example 3

Find the inverse of the matrix $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$.

Solution: The determinant of A is

$$\begin{aligned}
\det A &= 3 \times 4 - 6 \times 2 \\
&= 12 - 12 \\
&= 0.
\end{aligned}$$

Since $\det A = 0$,the matrix A does not have an inverse.

Exercise 1

Find if possible, the inverses of the following matrices:

a) $\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$ *b)* $\begin{bmatrix} 0 & 4 \\ 2 & 5 \end{bmatrix}$ *c)* $\begin{bmatrix} -2 & -3 \\ 6 & 9 \end{bmatrix}$ *d)* $\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}$.

Answers Exercise 1.

$$a)\begin{bmatrix} 3/2 & 1/2 \\ 2 & 1 \end{bmatrix} \quad b)\begin{bmatrix} -5/8 & 1/2 \\ 1/4 & 0 \end{bmatrix} \quad c)\text{ Inverse does not exist} \quad d)\ \frac{1}{11}\begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}.$$

Exercise 2

For

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix}$$

find A^{-1} and B^{-1} and show that $(AB)^{-1} = B^{-1}A^{-1}$.

Answers Exercise 2.

$$A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} -1/2 & 0 \\ 2 & 1 \end{bmatrix}.$$