UNIVERSITY

M5. Special Matrices

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is helpful to understand the definition of a number of different types of "special" matrices.

Transpose of a matrix

The transpose of a matrix $\mathbf{A}$ is denoted $\mathbf{A}^T$and is found by interchanging the rows and the columns.

The first row becomes the first column, the second row becomes the second column etc.

If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}^T$ is an $n \times m$ matrix.

Examples

1. If

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

then by interchanging the rows and columns we get

$$\mathbf{A}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}.$$

In this case $\mathbf{A}$ is a 3×2 matrix and its transpose $\mathbf{A}^T$ is a 2×3 matrix.

2. If

$$\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 7 & 5 \end{bmatrix}$$

then

$$\mathbf{B}^T = \begin{bmatrix} 1 & 7 \\ 2 & 5 \end{bmatrix}.$$

3. If

$$\mathbf{C} = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$$

then

$$\mathbf{C}^T = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}.$$

Note that in this case $\mathbf{C} = \mathbf{C}^T$ and $\mathbf{C}$ is called a symmetric matrix.

Symmetric Matrix

A symmetric matrix is a square matrix which is equal to its transpose. It is also symmetric about its leading diagonal (top left to bottom right).

Examples

1. The matrix

$$\mathbf{D} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$$

is symmetric because

$$\begin{aligned} \mathbf{D}^T &= \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \\ &= \mathbf{D}. \end{aligned}$$

2. The matrix

$$\mathbf{E} = \begin{bmatrix} 3 & 4 & 5 \\ 4 & -2 & -3 \\ 5 & -3 & 1 \end{bmatrix}$$

is symmetric because

$$\begin{aligned} \mathbf{E}^T &= \begin{bmatrix} 3 & 4 & 5 \\ 4 & -2 & -3 \\ 5 & -3 & 1 \end{bmatrix} \\ &= \mathbf{E}. \end{aligned}$$

3. The matrix

$$\mathbf{F} = \begin{bmatrix} 1 & -2 & 4 \\ -2 & 3 & -4 \\ 4 & -4 & 2 \end{bmatrix}$$

is symmetric because

$$\begin{aligned}\mathbf{F}^T &= \begin{bmatrix} 1 & -2 & 4 \\ -2 & 3 & -4 \\ 4 & -4 & 2 \end{bmatrix} \\ &= \mathbf{F}.\end{aligned}$$

Orthogonal Matrix

A square matrix is orthogonal if $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$ where $\mathbf{I}$ is the unit matrix (also called the identity matrix).

Because $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ it follows that for an orthogonal matrix $\mathbf{A}^T = \mathbf{A}^{-1}$. This can be useful for finding the inverse of an orthogonal matrix as it is usually easier to find the transpose than the inverse of a matrix.

Note that the determinant of an orthhogonal matrix is either equal to $+1$ (a rotation matrix) or -1 (a reflection matrix).

The rows of an orthogonal matrix are mutually orthogonal (perpendicular) unit vectors. The columns of an orthogonal matrix are also mutually orthogonal unit vectors.

Examples

The rotation matrix $\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix that rotates points, lines and regions through an angle of θ° anticlockwise. [1]

The determinant, $\det|\mathbf{A}| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1.$

$\mathbf{A}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\begin{aligned}\mathbf{A}^T\mathbf{A} &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

[1] Note that in the following sections, we use the trigonometric identity

$$\cos^2\theta + \sin^2\theta = 1.$$

and

$$\mathbf{A}\mathbf{A}^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

That is to say $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}$. Hence

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$= \frac{1}{1} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$= \mathbf{A}^T.$$

The reflection matrix $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is an orthogonal matrix that reflects points, lines and regions in the x-axis.

$|\mathbf{B}| = -1$ and $\mathbf{B}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\mathbf{B}^T\mathbf{B} = \mathbf{B}\mathbf{B}^T = \mathbf{I}$ and $\mathbf{B}^T = \mathbf{B}^{-1}$

Note that in this case $\mathbf{B}$ is both orthogonal and symmetric.

Exercise 1

Given

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & -3 \\ -3 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & 0 & 5 \\ 0 & -2 & 2 \\ 5 & 2 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 2 & -1 & 5 \\ 5 & -1 & 2 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

(a) Write down the transpose of each of the matrices.
(b) Which of the given matrices are symmetric?

Answers

a. $\mathbf{A}^T = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \mathbf{B}^T = \begin{bmatrix} 1 & -3 \\ -3 & 0 \end{bmatrix} \quad \mathbf{C}^T = \begin{bmatrix} 2 & 0 & 5 \\ 0 & -2 & 2 \\ 5 & 2 & 1 \end{bmatrix} \quad \mathbf{D}^T = \begin{bmatrix} 2 & 5 \\ -1 & -1 \\ 5 & 2 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

b.**B** and **C** are symmetric.

Exercise 2

Given

$$\mathbf{L} = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{N} = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

(a) Which of the given matrices are orthogonal? If the matrix is orthogonal, write down its inverse.

(b) Which of the given matices is a rotation matrix?

Answers

(a) **M**, **N** and **Q** are orthogonal

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{N}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{bmatrix} \quad \mathbf{Q}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(b) **M** is a rotation matrix. $|\mathbf{M}| = 1$.