IN3.4 Integration of Trigonometric Functions

$$\int \sin(ax+b)\,dx = \frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b)\,dx = -\frac{1}{a}\sin(ax+b) + c$$

$$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$$

This module deals with integration of trigonometric functions such as:

$$\int \sin(2x+3)\,dx$$
$$\int \cos(5x)\,dx$$
$$\int_1^2 \sec^2(x-2)\,dx.$$

Indefinite Integral (Antiderivative) of a Trigonometric Function

Recall that:

$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

It follows that indefinite integrals (or antiderivatives) of $\sin(x)$, $\cos(x)$ and $\sec^2(x)$ are of the form

$$\int \sin(x)\,dx = -\cos(x) + c$$
$$\int \cos(x)\,dx = \sin(x) + c$$
$$\int \sec^2(x)\,dx = \tan(x) + c$$

where c is a constant.

More general forms are: [1]

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$
$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$
$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$$

[1] These forms may be derived using integration by substitution. For example, let $u = ax+b$ then $du/dx = a$. Noting $\sin(ax+b) = 1 \times \sin(ax+b)$ and using substitution,

$$\begin{aligned}\int \sin(ax+b)\,dx &= \int \frac{1}{a}\sin(u)\,\frac{du}{dx}dx\\ &= \frac{1}{a}\int \sin(u)\,du\\ &= \frac{1}{a}(-\cos(u)) + c\\ &= -\frac{1}{a}\cos(ax+b\ ) + c.\end{aligned}$$

where a, b and c are constants.

Examples

1. $\int 5\cos(x)\,dx = 5\sin(x) + c, \ (a = 1, b = 0)$
2. $\int 3\cos(3-2x)\,dx = -\frac{3}{2}\sin(3-2x) + c \ (a = -2, \ b = 3)$
3. $\int 3\sin(3-x)\,dx = 3\cos(3-x) + c, \ (a = -1, \ b = 3)$
4. $\int \sec^2\left(\frac{x}{2}\right)dx = 2\tan\left(\frac{x}{2}\right) + c \ \left(a = \frac{1}{2}, \ b = 0\right)$.

Definite Integral of a Trigonometric Function

Now that we know how to get an indefinite integral (or antiderivative) of a trigonometric function we can consider definite integrals. To evaluate a definite integral we determine an antiderivative and calculate the difference of the values of the antiderivatve at the limits defined in the definite integral. For example consider

$$\int_0^{\pi/2} 3\cos(x)\,dx.$$

From the previous section we know an antiderivative is $3\sin(x) + c$ where c is a constant. The limits of the integral are 0 and $\pi/2$. So we have

$$\begin{aligned}
\int_0^{\pi/2} 3\cos(x)\,dx &= [3\sin(x) + c]_{x=0}^{x=\pi/2} && (1)\\
&= \left(3\sin\left(\frac{\pi}{2}\right) + c\right) - (3\sin(0) + c) && (2)\\
&= 3 + c - 0 - c && (3)\\
&= 3. && (4)
\end{aligned}$$

Note that the notation in line (1)

$$[3\sin(x) + c]_{x=0}^{x=\pi/2}$$

means substitute $x = \pi/2$ in the expression in brackets and subtract the expression in brackets evaluated at $x = 0$.

Note also that the constant c in lines (1) to (3) has no effect when evaluating a definite integral. Consequently we usually leave it out and write

$$\begin{aligned}
\int_0^{\pi/2} 3\cos(x)\,dx &= [3\sin(x)]_{x=0}^{x=\pi/2} && (1)\\
&= \left(3\sin\left(\frac{\pi}{2}\right)\right) - (3\sin(0)) && (2)\\
&= 3 - 0 && (3)\\
&= 3. && (4)
\end{aligned}$$

Examples

1. Evaluate $\int_0^{\pi/4} 5\cos(x)\,dx$.
   Solution: [2]

$$\begin{aligned}\int_0^{\pi/4} 5\cos(x)\,dx &= [5\sin(x)]_{x=0}^{x=\pi/4} \\ &= 5\sin\left(\frac{\pi}{4}\right) - 5\sin(0) \\ &= \frac{5}{\sqrt{2}}\end{aligned}$$

[2] Remember that $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

2. Evaluate $\int_{-\pi/4}^{\pi/4} 3\cos(\pi - 2x)\,dx$.
   Solution: [3]

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} 3\cos(\pi - 2x)\,dx &= \left[-\frac{3}{2}\sin(\pi - 2x)\right]_{x=-\pi/4}^{x=\pi/4} \\ &= \left(-\frac{3}{2}\sin\left(\pi - \frac{\pi}{2}\right)\right) - \left(-\frac{3}{2}\sin\left(\pi - 2\left(-\frac{\pi}{4}\right)\right)\right) \\ &= \left(-\frac{3}{2}\sin\left(\frac{\pi}{2}\right)\right) - \left(-\frac{3}{2}\sin\left(\pi + \frac{\pi}{2}\right)\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}(-1)\right) \\ &= -\frac{3}{2} - \frac{3}{2} \\ &= -3.\end{aligned}$$

[3] Remember that $\sin(\pi/2) = 1$ and $\sin(3\pi/2) = -1$.

3. Evaluate $\int_0^{\pi/6} 3\sin(\pi - 2x)\,dx$.
   Solution: [4]

$$\begin{aligned}\int_0^{\pi/6} 3\sin(\pi - 2x)\,dx &= \left[\frac{3}{2}\cos(\pi - 2x)\right]_{x=0}^{x=\pi/6} \\ &= \left(\frac{3}{2}\cos\left(\pi - 2\left(\frac{\pi}{6}\right)\right)\right) - \left(\frac{3}{2}\cos(\pi)\right) \\ &= \left(\frac{3}{2}\cos\left(\pi - \frac{\pi}{3}\right)\right) - \left(\frac{3}{2}(-1)\right) \\ &= \left(\frac{3}{2}\cos\left(\frac{2\pi}{3}\right)\right) + \frac{3}{2} \\ &= \frac{3}{2}\left(-\frac{1}{2}\right) + \frac{3}{2} \\ &= \frac{3}{4} \\ &= 0.75\end{aligned}$$

[4] Remember that $\cos(2\pi/3) = -1/2$ and $\cos(\pi) = -1$.

4. Evaluate $\int_0^{2\pi/3} \sec^2\left(\frac{x}{2}\right)\,dx$.
   Solution: [5]

[5] Remember that $\tan(0) = 0$ and $\tan(\pi/3) = \sqrt{3}$.

$$\begin{aligned}\int_0^{2\pi/3} \sec^2\left(\frac{x}{2}\right) dx &= \left[2\tan\left(\frac{x}{2}\right)\right]_{x=0}^{x=2\pi/3} \\ &= \left(2\tan\left(\frac{1}{2}\times\frac{2\pi}{3}\right)\right) - (2\tan(0)) \\ &= 2\tan\left(\frac{\pi}{3}\right) \\ &= 2\sqrt{3}.\end{aligned}$$

Exercises

1. Calculate:
   $a)\int \sec^2(4x)\,dx \quad b)\int 2\cos(1-x)\,dx \quad c) 2\sin\left(\frac{5-3x}{4}\right) dx$

2. Evaluate:
   $a)\int_0^{\pi/2} 3\cos(2x+\pi)\,dx \quad b)\int_{-\pi}^{0} 5\sin(x/2)\,dx \quad c)\int_0^{\pi/2} 2\sec^2(x/3) \quad dx$

Answers

1. $a)$ $\frac{1}{4}\tan(4x)+c$ $b)$ $-2\sin(1-x)+c$ $c)$ $\frac{8}{3}\cos\left(\frac{5-3x}{4}\right)+c$
2. $a)$ 0 $b)$ -10 $c)$ $2\sqrt{3}$