IN3.3 Integration of Exponential Functions

$$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$$

$$\int_0^2 e^{3x+1}\,dx = \left[\frac{1}{3}e^{3x+1}\right]_0^2 = \frac{1}{3}\left[e^7 - e\right]$$

This module deals with differentiation of exponential functions such as:

$$\int \exp(2x+3)\,dx$$
$$\int e^{3x}dx$$
$$\int_1^2 e^{x-1}dx.$$

Indefinite Integral of an Exponential Function

If $f(x) = e^x$ then $f'(x) = e^x$. Therefore an antiderivative (or indefinite integral) of e^x is e^x. That is

$$\int e^x dx = e^x + c, \text{ where } c \text{ is a constant.}$$

A more general form is: [1]

$$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c, \text{ where} a, b \text{ and } c \text{ are constants.}$$

[1] This form may be derived using integration by substitution. Let $u = ax+b$ then $du/dx = a$.Using substitution

$$\begin{aligned}\int e^{ax+b}dx &= \int \frac{1}{a}e^u \frac{du}{dx}dx \\ &= \frac{1}{a}\int e^u du \\ &= \frac{1}{a}e^u + c \\ &= \frac{1}{a}e^{ax+b} + c.\end{aligned}$$

Examples

1. $\int 2e^x dx = 2e^x + c$.
2. $\int e^{-5x+1}dx = -\frac{1}{5}e^{-5x+1} + c$, $(a = 5,\ b = 1)$.
3. $\int e^{\frac{x}{3}+4}dx = \frac{1}{1/3}e^{\frac{x}{s}+4} + c = 3e^{\frac{x}{3}+4} + c$, $\left(a = \frac{1}{3},\ b = 4\right)$.

Definite Integral of an Exponential Function

Now that we know how to get an antiderivative (or indefinite integral) of an exponential function we can consider definite integrals. To evaluate a definite integral we determine an antiderivative and calculate the difference of the values of the antiderivative at the limits defined in the definite integral. For example consider

$$\int_1^2 2e^x dx.$$

From the previous section we know an antiderivative is $2e^x + c$ where c is a constant. The limits of the integral are 1 and 2. So we have

$$\begin{aligned}
\int_1^2 2e^x dx &= [2e^x + c]_{x=1}^{x=2} && (1)\\
&= \left(2e^2 + c\right) - \left(2e^1 + c\right) && (2)\\
&= 2e^2 + c - 2e^1 - c && (3)\\
&= 2e^2 - 2e^1. && (4)
\end{aligned}$$

Note that the notation in line (1)

$$[2e^x + c]_{x=1}^{x=2}$$

means substitute $x = 2$ in the expression in brackets and subtract the expression in brackets evaluated at $x = 1$.

Note also that the constant c in lines (1) to (3) has no effect when evaluating a definite integral. Consequently we usually leave it out and write

$$\begin{aligned}
\int_1^2 2e^x dx &= [2e^x]_{x=1}^{x=2}\\
&= \left(2e^2\right) - \left(2e^1\right)\\
&= 2e^2 - 2e^1.
\end{aligned}$$

Examples

1. Evaluate $\int_{-1}^{4} 2e^x dx$.
   Solution:

$$\begin{aligned}
\int_{-1}^{4} 2e^x dx &= [2e^x]_{x=-1}^{x=4}\\
&= 2e^4 - 2e^{-1}.
\end{aligned}$$

2. Evaluate $\int_0^2 e^{-5x+1} dx$.

Solution:

$$\begin{aligned}\int_0^2 e^{-5x+1}dx &= \left[-\frac{1}{5}e^{-5x+1}\right]_{x=0}^{x=2}\\ &= \left(-\frac{1}{5}e^{-5(2)+1}\right) - \left(-\frac{1}{5}e^{-5(0)+1}\right)\\ &= \left(-\frac{1}{5}e^{-9}\right) - \left(-\frac{1}{5}e^{1}\right)\\ &= -\frac{1}{5}e^{-9} + \frac{1}{5}e\\ &= \frac{1}{5}\left(e - e^{-9}\right).\end{aligned}$$

3. Evaluate $\int_{-3}^{9} e^{\frac{x}{3}+4}dx$.
Solution:

$$\begin{aligned}\int_{-3}^{9} e^{\frac{x}{3}+4}dx &= \left[3e^{\frac{x}{3}+4}\right]_{x=-3}^{x=9}\\ &= \left(3e^{\frac{9}{3}+4}\right) - \left(3e^{\frac{-3}{3}+4}\right)\\ &= \left(3e^{3+4}\right) - \left(3e^{-1+4}\right)\\ &= 3\left(e^{7} - e^{3}\right).\end{aligned}$$

Exercises

1. Calculate:
$a)\int e^{3x}dx \quad b)\int e^{2-5x}dx \quad c)\int \frac{9e^{3x}+5}{e^{2x}}dx$ Hint: Divide through first.

2. Evaluate:
$a)\int_0^2 e^{3x}dx \quad b)\int_{-1}^{3} e^{2-5x}dx \quad c)\int_{-1}^{1} \frac{9e^{3x}+5}{e^{2x}} \quad dx$

Answers

1. $a)\ \frac{e^{3x}}{3}+c \qquad b)\ -\frac{e^{2-5x}}{5}+c \qquad c)\ 9e^{x}-\frac{5}{2e^{2x}}+c$

2. $a)\ \frac{e^{6}}{3}-\frac{1}{3} \qquad b)\ \frac{1}{5}\left(e^{7}-e^{-13}\right) \qquad c)\ 9\left(e-e^{-1}\right)-\frac{5}{2}\left(e^{2}-e^{-2}\right)$