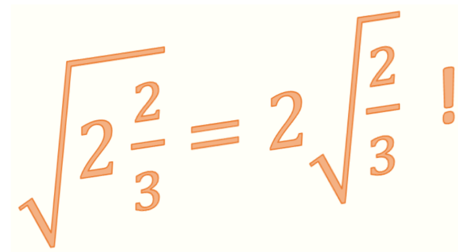
RMIT UNIVERSITY

ILS 3.1 *Surds*



A surd is an expression that cannot be expressed exactly without a square root, cube root or other root symbol. Because surds are irrational numbers, in decimal form they contain an infinite number of non-recurring digits.

Expressions like $\sqrt{2}$,$\sqrt{3}$, $\frac{1}{\sqrt{8}}$, $\sqrt{20}$,$\sqrt[3]{10}$,$\sqrt[4]{9}$, are all surds but $\sqrt{1}$, $\sqrt{9}$, $\frac{1}{\sqrt{100}}$, $\sqrt{25}$, $\sqrt[3]{64}$, $\sqrt[4]{16}$, are not because they can be written as $1, 3, 0.1, 5, 4$ and 2 respectively.

More complex expressions such as $3\sqrt{2}$ and $\sqrt{5} - \sqrt{2}$ are also surds.

Watch a short video here.

Simplifying Surds

Rules for simplifying surds

1. $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$
2. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

If a square root surd has a square factor greater than 1, that is, $4, 9, 16, 25, 36...$ then it can be simplified. Similarly if a cube root surd has a cube factor greater than 1, that is, $8, 27, 64, 125, \ldots$, then it can be simplified.

Examples

1. Simplify $\sqrt{200}$.

$$\begin{aligned}\sqrt{200} &= \sqrt{100 \times 2} \text{ (factorize out the square factor)}\\ &= \sqrt{100} \times \sqrt{2} \text{ (using Rule 1)}\\ &= 10\sqrt{2}.\end{aligned}$$

2. Simplify $\frac{\sqrt[3]{24}}{2}$.

$$\begin{aligned}\frac{\sqrt[3]{24}}{2} &= \frac{\sqrt[3]{8\times 3}}{2} \text{(factorize out the cube factor)}\\ &= \frac{\sqrt[3]{8}\times\sqrt[3]{3}}{2} \text{ (using Rule 1)}\\ &= \frac{2\sqrt[3]{3}}{2}\\ &= \sqrt[3]{3}.\end{aligned}$$

Addition and Subtraction of Surds

Only like surds can be added or subtracted.

Examples

1. Simplify $8\sqrt{5} - 3\sqrt{5}$.

$$8\sqrt{5} - 3\sqrt{5} = 5\sqrt{5} \text{ (just as } 8x - 5x = 5x.\text{)}$$

2. Simplify $2\sqrt{3} + 5\sqrt{7} - 10\sqrt{3}$.

$$2\sqrt{3} + 5\sqrt{7} - 10\sqrt{3} = 5\sqrt{7} - 8\sqrt{3} \text{ (just as } 2x + 5y - 10x = 5y - 8x.\text{)}$$

3. Simplify $\sqrt{18} - \sqrt{8} - \sqrt{20}$

$$\begin{aligned}\sqrt{18} - \sqrt{8} - \sqrt{20} &= \sqrt{9\times 2} - \sqrt{4\times 2} - \sqrt{4\times 5}\\ &= 3\sqrt{2} - 2\sqrt{2} - 2\sqrt{5}\\ &= \sqrt{2} - 2\sqrt{5}.\end{aligned}$$

Note that $\sqrt{2} - 2\sqrt{5}$ cannot be simplified because $\sqrt{2}$ and $\sqrt{5}$ are not like surds.

Multiplication of Surds

The following rule can be used to multiply terms containing surds:

$$a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}.$$

Examples

1. Simplify $\sqrt{2} \times \sqrt{3}$.

$$\sqrt{2} \times \sqrt{3} = \sqrt{6}.$$

2. Simplify $3\sqrt{3}\times 4\sqrt{5}$.

$$3\sqrt{3}\times 4\sqrt{5} = 12\sqrt{15}.$$

3. Simplify $2\sqrt{10}\times 7\sqrt{6}$.

$$\begin{aligned} 2\sqrt{10}\times 7\sqrt{6} &= 14\sqrt{60} \quad \text{(which can be simplified further)} \\ &= 14\sqrt{4\times 15} \\ &= 14\times 2\sqrt{15} \\ &= 28\sqrt{15} \end{aligned}$$

Division of Surds

The rule $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ may helpful in simplifying fractions that contain surds.

Examples

1. $\frac{\sqrt{18}}{\sqrt{6}} = \sqrt{\frac{18}{6}} = \sqrt{3}$.

2. $\frac{4}{\sqrt{2}} = \frac{\sqrt{16}}{\sqrt{2}} = \sqrt{\frac{16}{2}} = \sqrt{8} = \sqrt{2\times 4} = 2\sqrt{2}$.

Expansion of Brackets

The usual algebraic rules for removing brackets apply to brackets containing surds

1. $a\,(b+c) = ab + ac$

2. $(a+b)\,(c+d) = ac + bc + ad + bd$.

Examples

1. Simplify $\sqrt{2}(\sqrt{2}+5)$.

$$\sqrt{2}(\sqrt{2}+5) = 2 + 5\sqrt{2}$$

2. Simplify $\left(\sqrt{6}-4\sqrt{3})\right)\left(2\sqrt{2}-3\sqrt{5}\right)$.

$$\begin{aligned} \left(\sqrt{6}-4\sqrt{3}\right)\left(2\sqrt{2}-3\sqrt{5}\right) &= 2\sqrt{12} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15} \\ &= 2\sqrt{4\times 3} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15} \\ &= 2\times 2\times \sqrt{3} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15} \\ &= 4\sqrt{3} - 8\sqrt{6} - 3\sqrt{30} + 12\sqrt{15} \end{aligned}$$

Rationalizing Denominators

Sometimes fractions containing surds are required to be expressed with a rational denominator.

Examples

1. Rationalize the denominator of $\frac{2}{\sqrt{5}}$. [1]

$$\begin{aligned} \frac{2}{\sqrt{5}} &= \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} \\ &= \frac{2\sqrt{5}}{5} \end{aligned}$$

2. Rationalize the denominator of $\frac{\sqrt{5}}{3\sqrt{2}}$.

$$\begin{aligned} \frac{\sqrt{5}}{3\sqrt{2}} &= \frac{\sqrt{5}}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{10}}{6} \end{aligned}$$

[1] Note that

$$\frac{\sqrt{5}}{\sqrt{5}} = 1$$

and so multiplication by $\sqrt{5}/\sqrt{5}$ does not change the value of $2/\sqrt{5}$.

Conjugate Surds

The pair of expressions $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called conjugate surds. Each is the conjugate of the other. The product of two conjugate surds does NOT contain any surd term! That is,

$$\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b.$$

For example[2]

$$\begin{aligned} (\sqrt{10} + \sqrt{3})(\sqrt{10} - \sqrt{3}) &= \left(\sqrt{10}\right)^2 - \left(\sqrt{3}\right)^2 \\ &= 10 - 3 \\ &= 7. \end{aligned}$$

[2] Factorization of a difference of two squares: $(a+b)(a-b) = a^2 - b^2$.

We make use of this property of conjugates to rationalize denominators of the form $\sqrt{a} + \sqrt{b}$ or $\sqrt{a} - \sqrt{b}$.

Example

Rationalize the denominator of

$$\frac{\sqrt{3}}{5 + \sqrt{2}}.$$

Solution:

$$\begin{aligned}\frac{\sqrt{3}}{5+\sqrt{2}} &= \frac{\sqrt{3}}{5+\sqrt{2}} \times \frac{5-\sqrt{2}}{5-\sqrt{2}}\\ &= \frac{\sqrt{3}\left(5-\sqrt{2}\right)}{\left(5+\sqrt{2}\right)\left(5-\sqrt{2}\right)}\\ &= \frac{5\sqrt{3}-\sqrt{6}}{5^2-\left(\sqrt{2}\right)^2}\\ &= \frac{5\sqrt{3}-\sqrt{6}}{25-2}\\ &= \frac{5\sqrt{3}-\sqrt{6}}{23}\end{aligned}$$

Try Exercise 1 now.

Adding and Subtracting Fractions Involving Surds

When adding or subtracting fractions containing surds it is generally advisable to first rationalize the denominators of each fraction.

Example

Evaluate $\frac{2}{3\sqrt{2}+1} + \frac{1}{\sqrt{3}-\sqrt{2}}$.
Solution:

$$\begin{aligned}\frac{2}{3\sqrt{2}+1} + \frac{1}{\sqrt{3}-\sqrt{2}} &= \frac{2}{3\sqrt{2}+1} \times \frac{3\sqrt{2}-1}{3\sqrt{2}-1} + \frac{1}{\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} \text{(rationalize denominators)}\\ &= \frac{6\sqrt{2}-2}{18-1} + \frac{\sqrt{3}+\sqrt{2}}{3-2}\\ &= \frac{6\sqrt{2}-2}{17} + \frac{\sqrt{3}+\sqrt{2}}{1}\\ &= \frac{6\sqrt{2}-2}{17} + \frac{17\left(\sqrt{3}+\sqrt{2}\right)}{17} \text{(obtain a common denominator to add the fractions)}\\ &= \frac{6\sqrt{2}-2+17\left(\sqrt{3}+\sqrt{2}\right)}{17}\\ &= \frac{23\sqrt{2}-2+17\sqrt{3}}{17}.\end{aligned}$$

Try Exercise 2 now.

Exercise 1

Simplify

1. $\sqrt{500}$
2. $\frac{2}{\sqrt{20}}$
3. $\sqrt{2}+\sqrt{7}+3\sqrt{2}-4\sqrt{7}$
4. $\sqrt{54}-\sqrt{24}$
5. $\sqrt{10}\times 3\sqrt{10}$
6. $2\sqrt{8}\times 2\sqrt{50}\times\sqrt{2}$
7. $\frac{\sqrt{32}}{\sqrt{2}}$
8. $\frac{4\sqrt{2}\times 3\sqrt{3}}{6\sqrt{6}}$

Answers:

1. $10\sqrt{5}$ 2. $\frac{1}{\sqrt{5}}$ 3. $4\sqrt{2}-3\sqrt{7}$ 4. $\sqrt{6}$
5. 30 6. $80\sqrt{2}$ 7. 4 8. 2.

Exercise 2

A) Expand and simplify if possible

1. $\sqrt{2}\left(\sqrt{2}-8\right)$
2. $\left(\sqrt{11}+3\right)\left(\sqrt{11}-3\right)$

Answers

1. $2-8\sqrt{2}$ 2. 2.

B) Express with a rational denominator

1. $\frac{1}{\sqrt{10}}$
2. $\frac{\sqrt{3}+2}{\sqrt{3}-2}$

Answers

1. $\frac{\sqrt{10}}{10}$ 2. $-7-4\sqrt{3}$.

C) Evaluate and express with a rational denominator

$\frac{2}{\sqrt{3}-1}+\frac{3}{2-\sqrt{3}}$

Answer

$4\sqrt{3}+7$.