ILS 2.1 Logarithms

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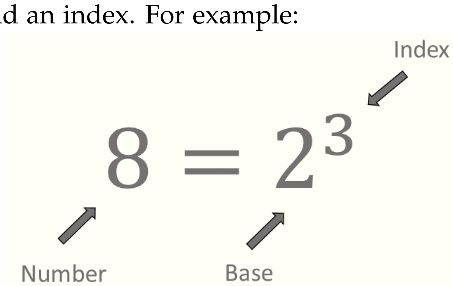
The modeling of growth and decay in areas such as finance, epidemiology and science makes use of equations with logarithms and exponentials. For example the equation $\log_e \frac{N}{N_0} \approx -\lambda t$ is used to describe radioactive decay. The laws for working with logarithms enable us to solve equations that cannot be solved with other algebraic techniques.

You can watch a short video here.

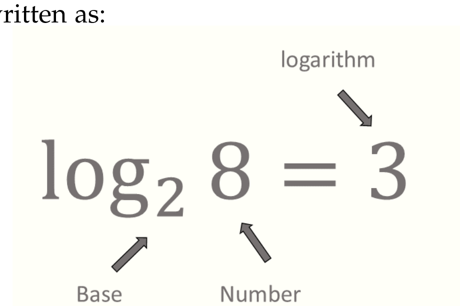
Definition

The logarithm of a number is the power to which the base must be raised to produce the number. This means that a logarithm (or log) is an index. Every index or log expression contains a number, a base and an index. For example:



And we say " eight is equal to two to the power of three".

This statement has an equivalent logarithmic form and may be written as:



We say "the log to base two of eight is equal to three" (the power of two that gives eight is three).

Generally, providing $a > 0$, $n > 0$ and $a \neq 1$,

$$a^x = n \Leftrightarrow \log_a n = x$$

Logarithms and the Calculator

While any positive number except one can be the base of a logarithm, 10 and e are most common.

To find the power of 10 that produces 50 we use the LOG button on the calculator: $\log_{10} 50 \approx 1.7$ and $10^{1.7} \approx 50$.

To find the power of e that produces 50 we use the LN button on the calculator: $\log_e 50 = \ln 50 \approx 3.9$ and $e^{3.9} \approx 50$.

Examples

1. $5^3 = 125 \Leftrightarrow \log_5 125 = 3$
2. $3^{-2} = \frac{1}{9} \Leftrightarrow \log_3 \frac{1}{9} = -2$
3. $25^{\frac{1}{2}} = 5 \Leftrightarrow \log_{25} 5 = \frac{1}{2}$

Evaluating Logarithms

To evaluate a logarithm it may be helpful to create an equation and change to index form.

1. Evaluate $\log_5 125$

$$\begin{aligned} \text{Let} \log_5 125 &= x \\ 5^x &= 125 \text{ (change to index form)} \\ 5^x &= 5^3 \text{ (express 125 as a power of 5)} \\ x &= 3 \text{ (equating indices).} \end{aligned}$$

2. Evaluate $\log_{10} 0.0001$

$$\begin{aligned} \text{Let} \log_{10} 0.0001 &= x \\ 10^x &= 0.0001 \text{ (change to index form)} \\ 10^x &= 10^{-4} \text{ (express 0.0001 as a power of 10)} \\ x &= -4 \text{ (equating indices).} \end{aligned}$$

Three Important Properties of Logarithms

- For $a > 0$,

$$a = 1 \Leftrightarrow \log_a 1 = 0.$$

That is, for any $a > 0$, the log of 1 is zero.

- For $a > 0$

$$a = a^1 \Leftrightarrow \log_a a = 1.$$

- If $\log_a m = \log_a p$ then $m = p$.

Logarithm Laws

Corresponding to the three index laws, there are three laws of logarithms to help in manipulating logarithms. If $m, n > 0$ and $a > 0,\ a \neq 1$ then

1. First Logarithm Law

$$\log_a(mn) = \log_a m + \log_a n$$

2. Second Logarithm Law

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

3. Third Logarithm Law

$$\log_a m^p = p \log_a m$$

The laws can be used to simplify and evaluate logarithmic expressions

Examples

1. $\log_a 15 = \log_a(5 \times 3) = \log_a 5 + \log_a 3$
2. $\log_a \frac{12}{5} = \log_a 12 - \log_a 5$
3. $\log_a \left(9^2\right) = 2 \log_a 9 = 2 \log_a 3^2 = 4 \log_a 3$
4. $\log_{10} 6 + \log_{10} 2 = \log_{10}(6 \times 2) = \log_{10} 12$
5. $\log_3 6a + \log_3 b - \log_3 2a = \log_3 \left(\frac{6a \times b}{2a}\right) = \log_3 3b$
6. Simplify $\frac{1}{2} \log_{10} 36 - \log_{10} 15 + 2 \log_{10} 5$

$$\begin{aligned}
\frac{1}{2} \log_{10} 36 - \log_{10} 15 + 2 \log_{10} 5 &= \log_{10} 36^{\frac{1}{2}} - \log_{10} 15 + \log_{10} 5^2 \\
&= \log_{10} 6 - \log_{10} 15 + \log_{10} 25 \\
&= \log_{10} \frac{6 \times 25}{15} \\
&= \log_{10} 10 \\
&= 1
\end{aligned}$$

Logarithmic Equations

The laws of logarithms can be used to solve equations involving logarithms or in which the variable is a power.

Examples

1. Solve for x if $\log(2x+1) = \log_{10} 3$

$$\begin{aligned} \log_{10}(2x+1) &= \log_{10} 3 \\ 2x+1 &= 3 \\ x &= 1 \end{aligned}$$

2. Solve for x if $2^{x-3} = 10$

$$\begin{aligned} 2^{x-3} &= 10 \\ \log_{10} 2^{x-3} &= \log_{10} 10 \text{ (take logs of both sides)} \\ (x-3)\log_{10} 2 &= \log_{10} 10 \\ (x-3)(0.301) &= 1 \text{ (use a calculator to find } \log_{10} 2) \\ x-3 &= \frac{1}{0.301} \\ x &= 3 + \frac{1}{0.301} \\ x &= 6.32 \end{aligned}$$

Exercise 1

Write in logarithm form

a) $3^2 = 9$
b) $10^4 = 10000$
c) $10^{-2} = 0.01$
d) $e^a = b$

Answers

$a)$ $\log_3 9 = 2$ $\quad b)$ $\log_{10} 10000 = 4$ $\quad c)$ $\log_{10} 0.01 = -2$ $\quad d)$ $\log_e b = a$.

Exercise 2

Evaluate without using a calculator

a) $\log_7 49$
b) $\log_{10} \sqrt{10}$
c) $\log_5 1$

d) $\log_{10} 100000$
e) $\log_5 5$

Answers

$a)\, 2 \quad b)\, 1/2 \quad c)\, 0 \quad d)\, 5 \quad e)\, 1$

Exercise 3

Simplify
a) $\log_4 8 + \log_4 3 - \log_4 2$
b) $\frac{1}{2}\log_{10} 25 - \log_{10} 4 + 2\log_{10} 3$
c) $\log_e e^2 + 2\log_e 3 - \log_e 18$
d) $\log_a 4 + 2\log_a 3 – 2\log_a 6$
e) $\frac{1}{2}\log_{10} a^2 + 3\log_{10} b - \log_{10} 3ab^2$

Answers

$a)\ \log_4 12 \quad b)\ \log_{10}(45/4) \quad c)\, 2 - \log_e 2 \quad d)\, 0 \quad e)\ \log_{10} b - \log_{10} 3$

Exercise 4

Solve for x
a) $\log_{10} x = \log_{10} 4 - \log_{10} 2$
b) $\log_2 2x - 2\log_2 3 = \log_2 6$
c) $10^{m+1} = 7$

Answers

$a)\, x = 2 \quad b)\, x = 27 \quad c)\, m = -0.155$