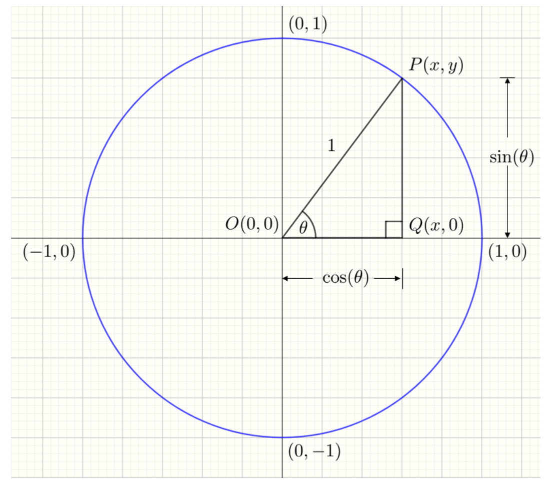
# *FG6: Circular Functions*

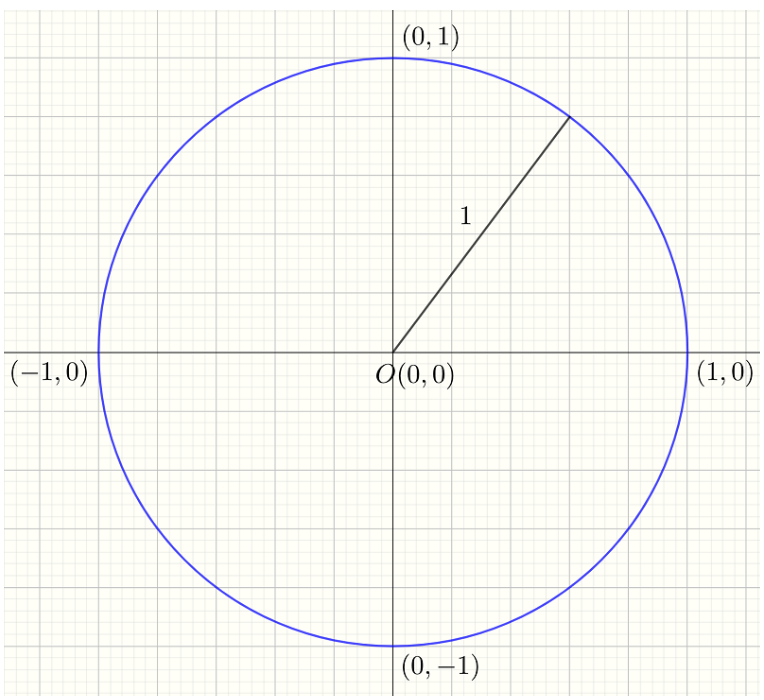



Trigonometric functions such as sin, cos and tan are usually defined as the ratios of sides in a right angled triangle. This module defines the trigonometric functions using angles in a unit circle.

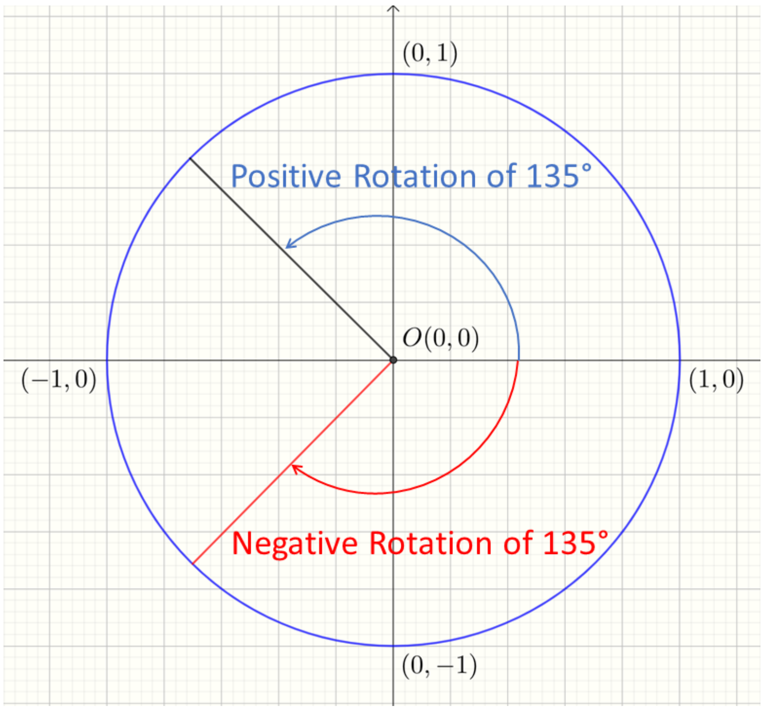
## *The Unit Circle*

A unit circle is a circle that is centered at the origin[1] and has a radius of one as shown below.

[1] The origin is the point with coordinates $(0, 0)$



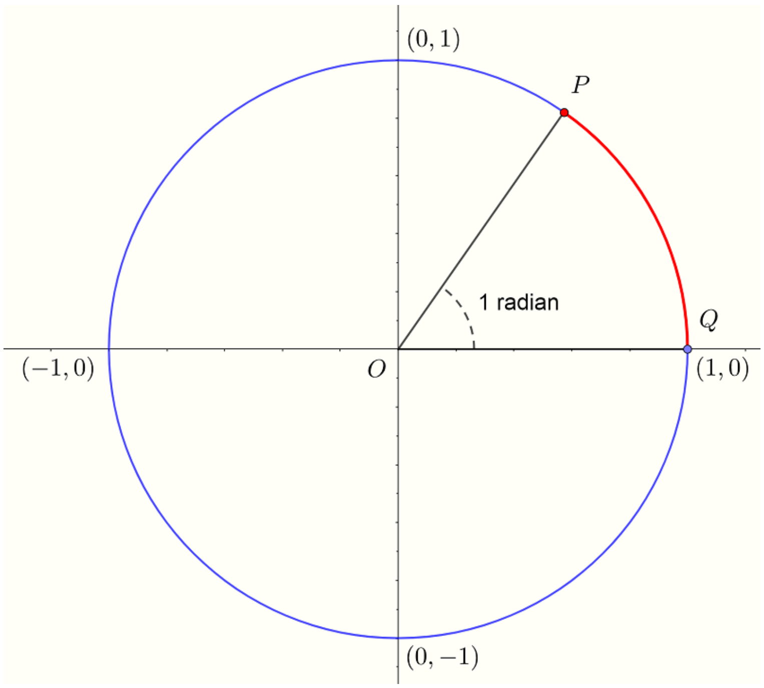

An angle measured from the positive $x-$ axis maybe used to define any point on the unit circle. Angles may be positive or negative. Positive angles are in an anti-clockwise direction. and negative angles are in a clockwise direction as shown below.




Although there are $360^\circ$ in a circle, it is possible to to rotate through more than $360^\circ$. For example a rotation of $45^\circ$ identifies the same point as a rotation of $45^\circ + 360^\circ = 405^\circ$.

## *Angular Measurement and the Unit Circle*

Though angles are often measured in degrees, they may also be measured in radians. One radian is the angle subtended from the $x-$ axis by an arc whose length is equal to the radius of the circle. In a unit circle , the radius is 1. This is illustrated below:




In the figure, the red arc is of length 1 and is the same length as the radius. The angle formed is one radian. Radians are abbreviated as rad or a superscript $c$ like $^c$.

Since the circumference of the unit circle is $2\pi$ radians[2] and there are $360°$ in a circle we have

[2] The circumference formula is $C = 2\pi r$ where $r$ is the radius. Since $r = 1$ in a unit circle, we see the circumference is $2\pi$.

$$360° = 2\pi \text{ rad}.$$

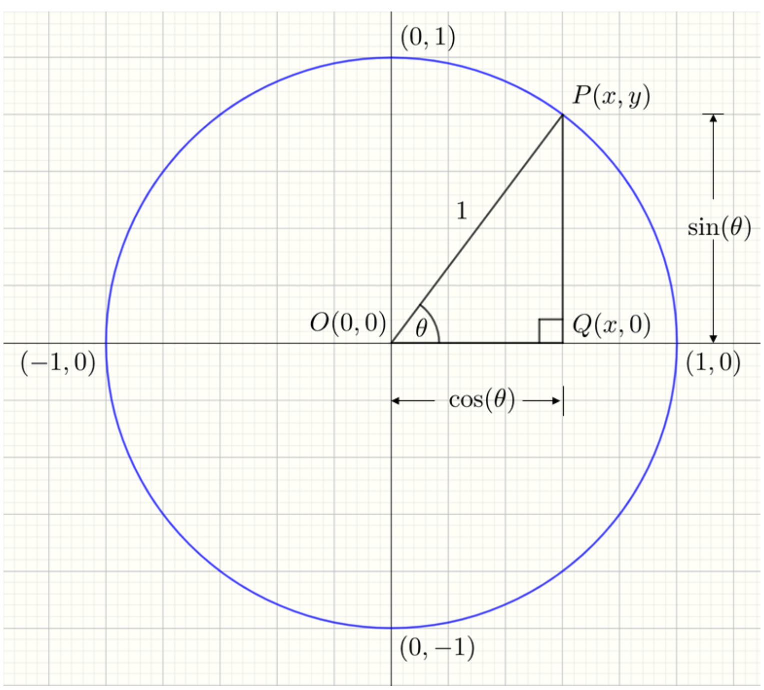
By dividing the above equation by a number it is possible to derive the following angles in degrees in terms of radians as follows:

$$\begin{aligned}
30^\circ &= \frac{\pi}{6} \\
45^\circ &= \frac{\pi}{4} \\
60^\circ &= \frac{\pi}{3} \\
90^\circ &= \frac{\pi}{2} \\
180^\circ &= \pi \\
150^\circ &= 5 \times 30^\circ \\
&= 5 \times \frac{\pi}{6} \\
&= \frac{5\pi}{6} \\
270^\circ &= 3 \times 90^\circ \\
&= 3 \times \frac{\pi}{2} \\
&= \frac{3\pi}{2}.
\end{aligned}$$

Angles in radians may also be positive and negative. The same convention is used as for degrees, a positive rotation is anti-clockwise and a negative rotation is clockwise.

## *Sine and Cosine on the Unit Circle*

Imagine a point $P(x, y)$ on the unit circle as shown below.



In the triangle $POQ$, the hypotenuse $OP$ is 1, the adjacent side to the angle $\theta$ is $OQ$ and the opposite side to the angle $\theta$ is $QP$. Using the definitions[3] of sine and cosine in a right angled triangle we get:

[3] Remember that

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

and

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}.$$

$$\begin{aligned}\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}}\\ &= \frac{OQ}{OP}\\ &= \frac{x}{1}\\ &= x\end{aligned}$$

and

$$\begin{aligned}\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}}\\ &= \frac{QP}{OP}\\ &= \frac{y}{1}\\ &= y\end{aligned}$$

That is, any point on the unit circle has coordinates

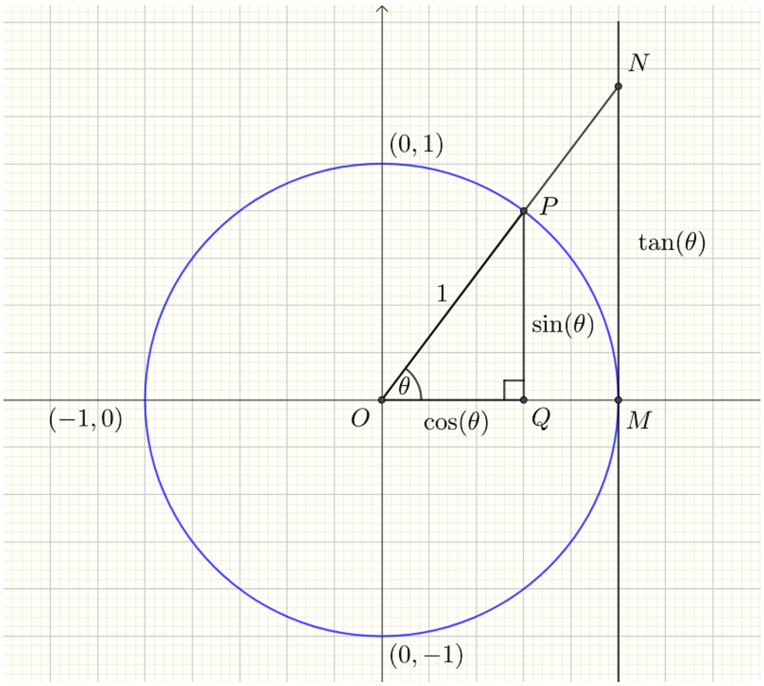
$$\begin{aligned}x &= \cos\theta\\ y &= \sin\theta\end{aligned}$$

where $\theta$ is the angle (positive or negative) measured from the positive $x-$ axis to the point.

Examples: $\sin(0) = 0$, $\sin(90^\circ) = \sin\left(\frac{\pi}{2}\right) = 1$, $\cos(0) = 1$, $\cos\left(\frac{\pi}{2}\right) = \cos(90^\circ) = 0$.

## *The Tangent Function*

Consider the figure below.




The triangles $OMN$ and $OQP$ are similar therefore

$$\frac{MN}{OM} = \frac{PQ}{OQ}$$
$$MN = \frac{PQ}{OQ} \qquad \textit{since } OM = 1$$

$OM = 1$ (it is a unit circle). We define the length $MN$ to be the tangent[4] of the angle $\theta$ and abbreviate this to $\tan(\theta)$. Also $\sin(\theta) = PQ$ and $\cos(\theta) = OQ$. Substituting these relationships into equation (1) we have

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

This agrees with the usual definition of tan from right angled trigonometry.

[4] We call this the tangent as it is the distance measured on the vertical tangent to the circle that passes through the point $(1, 0)$.

## *Exact Values of Sine, Cosine and Tangent Functions*

To find the value of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ we usually use a scientific calculator[5]. However, there are some special angles that are worth remembering. These are shown in the following table.

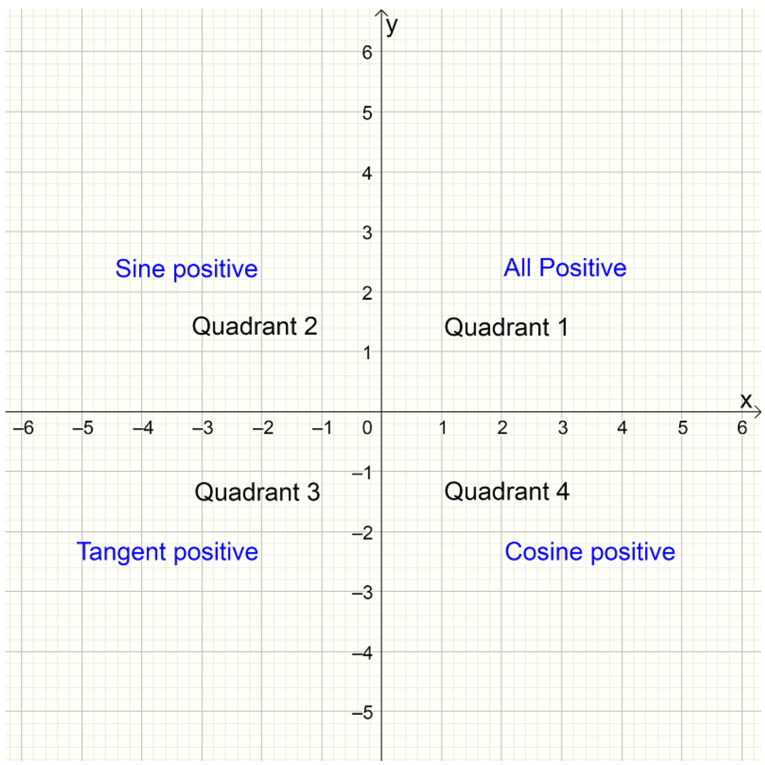
[5] When using the calculator it is essential that it be in degrees or radians mode.

| Angle $(\theta)$ | $0°$ | $30° = \pi/6$ | $45° = \pi/4$ | $60° = \pi/3$ | $90° = \pi/2$ | $180° = \pi$ | $270° = \frac{3\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $\sin(\theta)$ | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 | -1 |
| $\cos(\theta)$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | -1 | 0 |
| $\tan(\theta)$ | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | undefined | 0 | undefined |

If you know the exact values above, you can work out the exact values for other angles using the symmetry of the unit circle.

## *Quadrants*

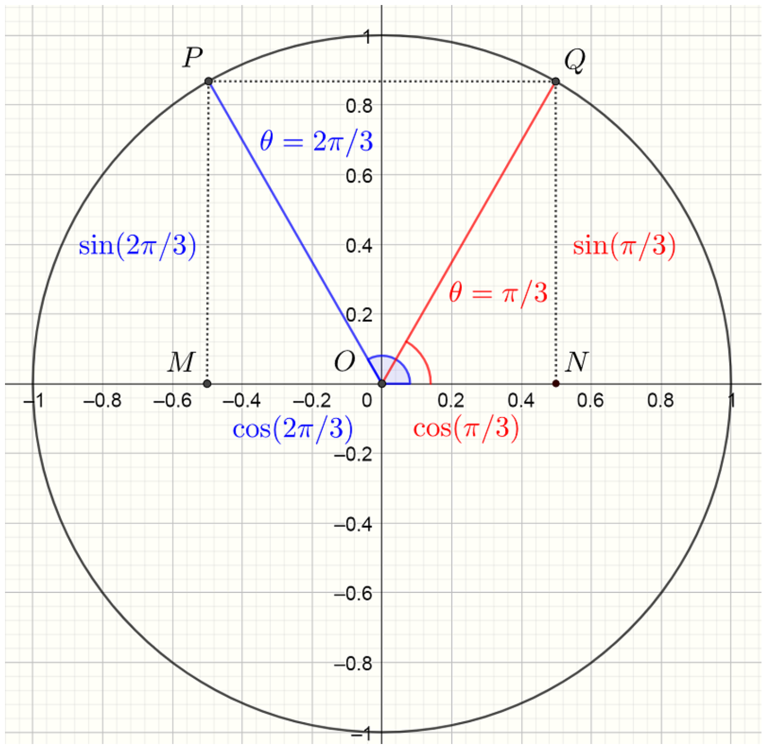
The coordinate plane is divided into an upper and lower section by the $x$-axis. It is further divided into quadrants by the $y$-axis. These four quadrants are numbered from one to four in an anti-clockwise direction as shown below:




Knowing exact values for some angles in quadrant 1, allows you to find exact values in other quadrants.

*Example 1: Find the exact value of* $\sin(2\pi/3)$ *and* $\cos(2\pi/3)$.

Solution: It is important to make a sketch as shown below. Remember $\frac{2\pi}{3} = 120°$.




First draw in the angle you are interested in. In this case it is $2\pi/3$ and is shown in blue as are the required values for the circular functions sin and cos. Now use symmetry of the unit circle about the $y-$ axis to draw in an angle that is related to $2\pi/3$ but for which you know the exact value. In this case it is $\pi/3$ and is shown in red. You can see that the length of $NQ$ is the same as $MP$ which means

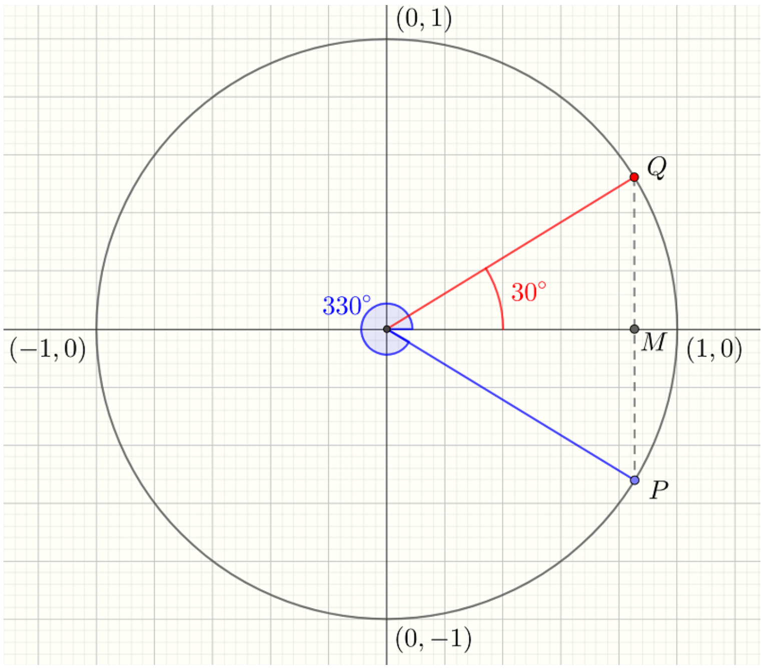
$$\begin{aligned} NQ &= MP \\ \sin\left(\frac{2\pi}{3}\right) &= \sin\left(\frac{\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2} \quad \text{from the table.} \end{aligned}$$

Now note that the the lengths of $ON$ and $MO$ are the same but $MO = -ON = -\cos(\pi/3)$ and so

$$\begin{aligned} \cos\left(\frac{2\pi}{3}\right) &= -ON \\ &= -\cos(\pi/3) \\ &= -\frac{1}{2} \quad \text{from the table.} \end{aligned}$$

*Example 2: Find the exact value of* $\sin(330°)$.

Plotting $330°$ on a unit circle shows that $\sin(330°)$ is closely related to $\sin(30°)$. The $y-$coordinates differ only by sign because the distances from the $x-$axis are the same.




We have

$$\begin{aligned} MP &= -MQ \\ \sin(330°) &= -\sin(30°) \\ &= -\frac{1}{2} \quad \text{from the table.} \end{aligned}$$

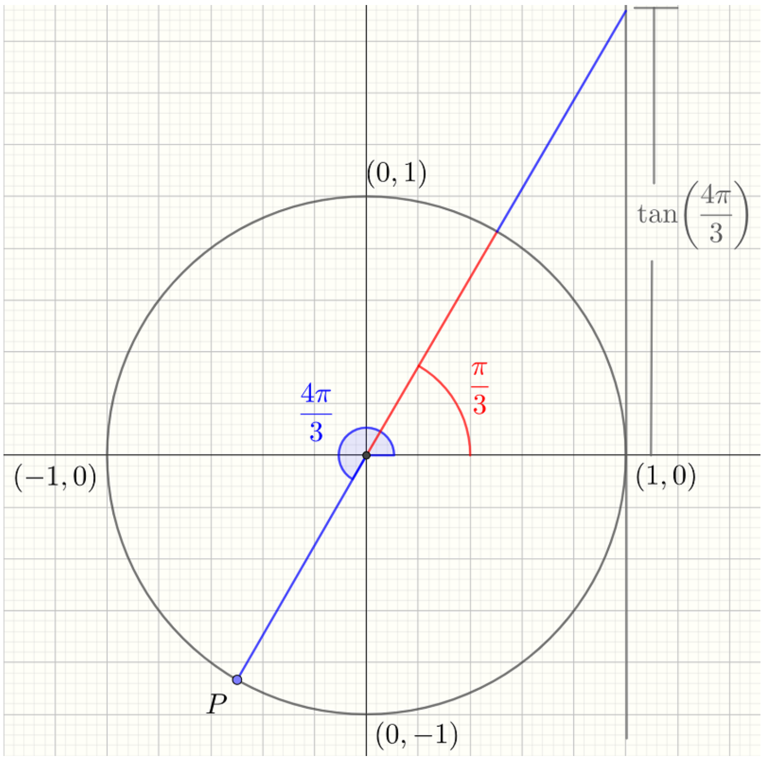
Note:

$$\begin{aligned} \cos(330°) &= \cos(360° - 30°) \\ &= \cos(30°) \\ &= \frac{\sqrt{3}}{2} \quad \text{from the table.} \end{aligned}$$

*Example 3: Find the exact value of* $\tan(4\pi/3)$.

Plotting $4\pi/3$ on a unit circle shows that $\tan(4\pi/3)$ is closely related to $\tan(\pi/3)$.

Remember $\frac{4\pi}{3} = \frac{3\pi}{3} + \frac{\pi}{3} = \pi + \frac{\pi}{3}$




We see that

$$\tan\left(\frac{4\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right)$$
$$= \sqrt{3} \quad \text{from the table.}$$

*Exercises*

1) What are the coordinates for points on the unit circle that make the following angles with the positive $x-$axis?

$a)$ $30°$ $b)$ $125°$ $c)$ $-60°$
$d)$ $270°$ $e)$ $-180°$ $f)$ $720°$

2) Find exact values for:

$a)$ $\sin(330°)$ $b)$ $\cos(210°)$ $c)$ $\sin(-30°)$
$d)$ $\cos(90°)$ $e)$ $\tan(300°)$ $f)$ $\cos(180°)$
$g)$ $\sin(-120°)$ $h)$ $\cos(315°)$

*Answers*

1)

$a)$ $(cos(30°), sin(30°)) = (0.87, 0.5)$ $b)$ $(-0.56, 0.82)$ $c)$ $(0.5, -0.87)$ $d)$ $(0, -1)$ $e)$ $(-1, 0)$ $f)$ $(1, 0)$

2)

$a)\,0.5$ $b)\,\sqrt{3}/2$ $c)\,-0.5$ $d)\,0$ $e)\,\sqrt{3}$ $f)\,-1$
$g)\,-\sqrt{3}/2$ $h)\,1/\sqrt{2}$