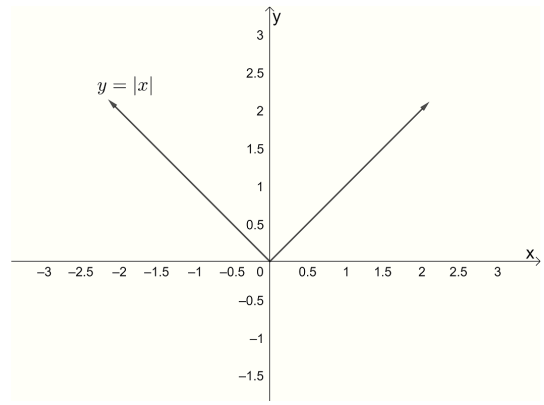
FG4: The Absolute Value Function



Introduction

The absolute value of a number gives a measure of its size or magnitude regardless of whether it is positive or negative.

If a number is plotted on a number line then its absolute value can be considered to be the distance from zero.

The absolute value of a number or a pro-numeral is designated by two vertical lines such as $|\cdot|$. For example the absolute value of the pro-numeral x is $|x|$.

Examples

1. $|2| = 2$

2. $|-2| = 2$

3. $|-4+3| = |-1| = 1$

4. $|-8| + |-1| = 8 + 1 = 9$

5. $|x| = 7 \Rightarrow x = 7$ or $x = -7$

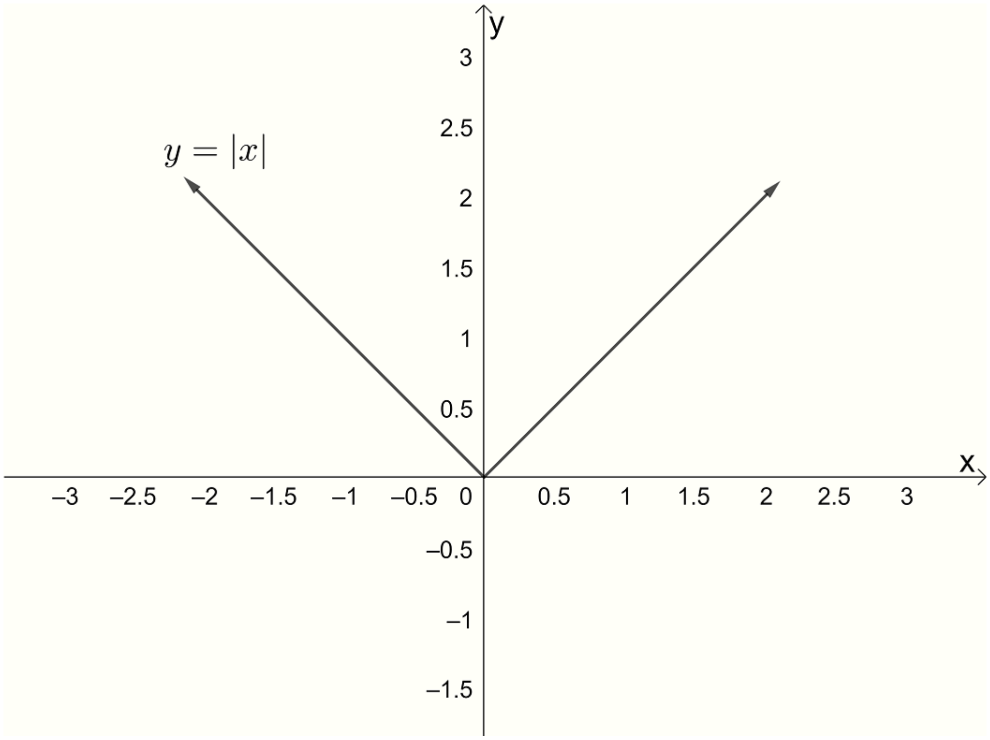
The Absolute Value Function and its Graph

The absolute value function is a hybrid function[1] defined as follows:[2]

$$f : \mathbb{R} \to \mathbb{R}, \text{where } f(x) = |x| = \begin{cases} -x, & x < 0 \\ x & x \geq 0 \end{cases}$$

with graph

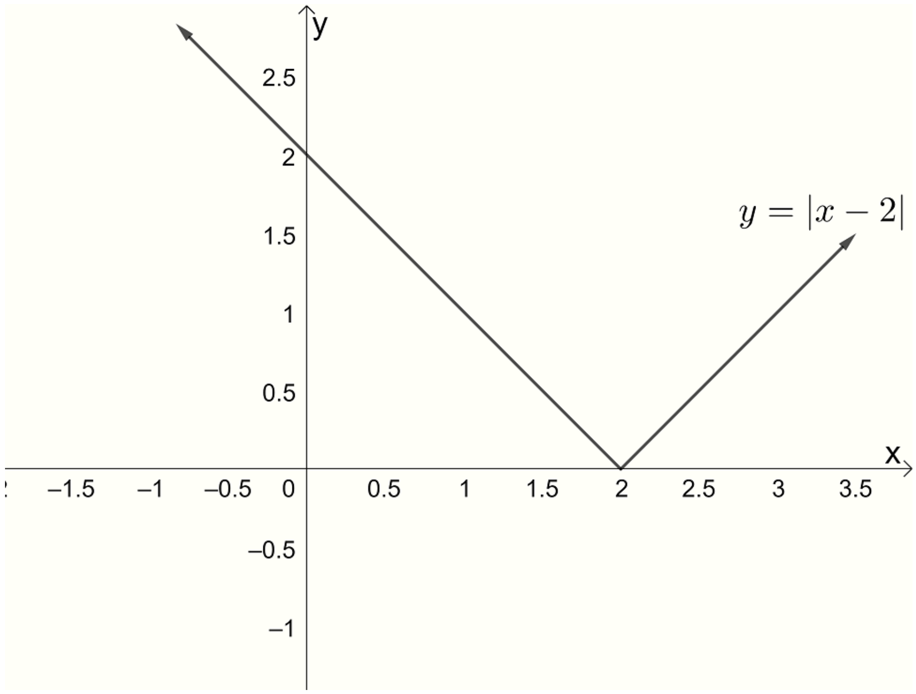
[1] A hybrid function involves two or more cases. Each case depends on the domain of the function.
[2] In what follows, $\mathbb{R}$ is the set of real numbers.



The domain of $f(x)$=$|x|$ is $\mathbb{R}$ and the range of $f(x)$ is $\mathbb{R}^+ \cup \{0\}$. That is the set of all positive real numbers and zero.

The graph of $y = |x|$ may be translated in the same way as the graphs of other functions. Compare the graphs of the following functions with that of $y = |x|$

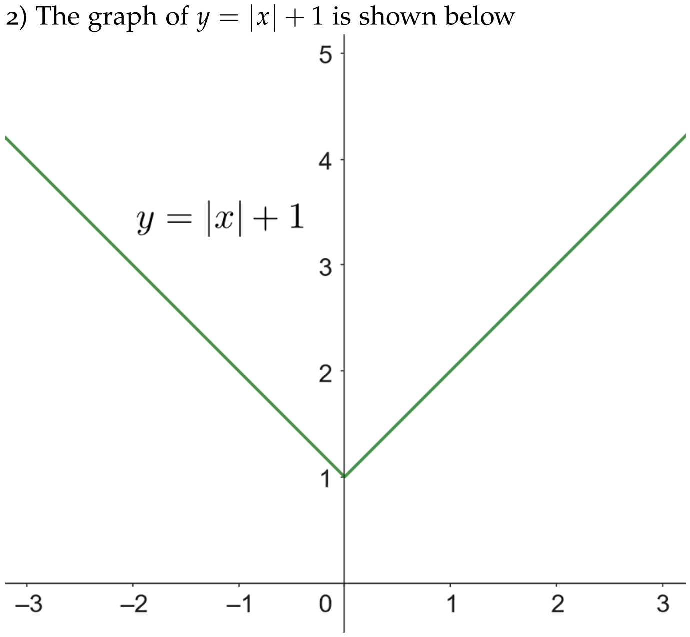
1) The graph of $y = |x - 2|$ is shown below



and is the graph of $y = |x|$ translated horizontally two units to the right. [3]

[3] The graph of $y = |x+2|$ is the graph of $y = |x|$ shifted two units to the left.

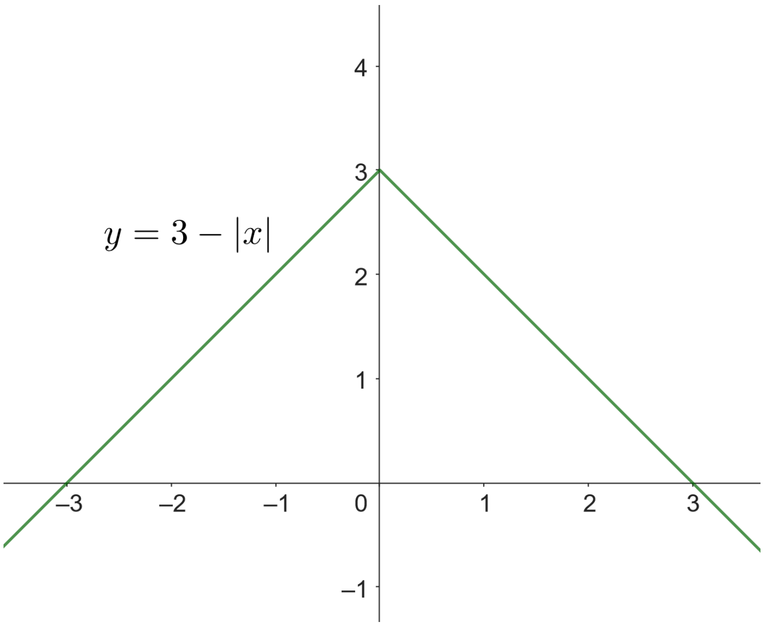
2) The graph of $y = |x| + 1$ is shown below



and is the graph of $y = |x|$ translated vertically one unit up. [4]

[4] The graph of $y = |x| - 1$ is the graph of $y = |x|$ translated vertically one unit down.

3) The graph of $y = 3 - |x| = -|x| + 3$ is shown below

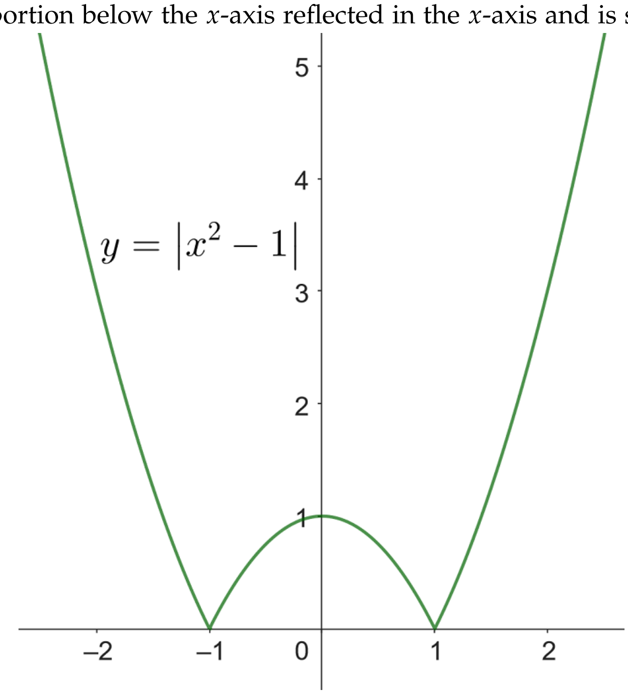


and is the graph of $y = |x|$ reflected in the x axis followed by a vertical shift of three units up.

In general, to sketch the graph of $y = |f(x)|$, we need to sketch the graph of $y = f(x)$ first and then reflect in the x-axis the portion of the graph which is below the x-axis.

4) Sketch $\{(x, y) : y = |x^2 - 1|\}$

The graph of this function is the graph of $y = x^2 - 1$ with the portion below the x-axis reflected in the x-axis and is shown below:



Equations and Inequalities Involving $|f(x)|$

Because $y = |f(x)|$ is a hybrid function, two cases must be considered when solving equations and inequalities.

Examples

1) Solve $|x-2| = 3$

Solution:

If $|x-2| = 3$ we must consider the two cases:

$$\begin{aligned} x-2 &= 3 \\ x &= 3+2 \\ &= 5 \end{aligned}$$

and

$$\begin{aligned} x-2 &= -3 \\ x &= -3+2 \\ &= -1. \end{aligned}$$

Hence the answer is $x = -1$ and $x = 3$.

2) Solve $|2x+1| = |x-5|$.

Solution:

With an absolute value expression on each side of the equation it is easier to square both sides.

$$\begin{aligned} |2x+1| &= |x-5| \\ (2x+1)^2 &= (x-5)^2 \\ 4x^2+4x+1 &= x^2-10x+25 \\ 4x^2+4x+1-x^2+10x-25 &= 0 \\ 3x^2+14x-24 &= 0 \\ (3x-4)(x+6) &= 0 \end{aligned}$$

So

$$\begin{aligned} 3x-4 &= 0 \\ x &= \frac{4}{3} \end{aligned}$$

or

$$\begin{aligned} x+6 &= 0 \\ x &= -6. \end{aligned}$$

Hence the answer is $x = 4/3$ and $x = -6$.

3) Find the set of $x \in \mathbb{R}$ such that $\left|\frac{2-x}{3}\right| < 4$.

Solution: [5]

We have

$$\left|\frac{2-x}{3}\right| < 4.$$

Multiplying each side by 3 :

$$|2-x| < 12$$

and so

$$-12 < 2-x < 12.$$

Adding 2 to all sides we get:

$$-14 < -x < 10.$$

Multiplying by -1, and noting the reversal of the inequality signs,

$$14 > x > -10 \text{ or } -10 < x < 14.$$

Hence the answer is that x is greater than -10 but less than 14. More formally, this may be expressed as a set

$$\{x \in \mathbb{R} : -10 < x < 14\}.$$

[5] Care must be taken when multiplying or dividing an inequality by a negative number. In such cases the inequality is reversed.
The answer to this type of question is in fact a set as it involves an infinite number of solutions.

4) Find the set of $x \in \mathbb{R}$ such that $\left|\frac{x-2}{3}\right| \geq 2$.

Solution:

Multiply both sides by 3 to get

$$|x-2| \geq 6.$$

Hence

$$\begin{aligned} x-2 &\geq 6 \\ x &\geq 8 \end{aligned}$$

or

$$\begin{aligned} x-2 &\leq -6 \\ x &\leq -4. \end{aligned}$$

Hence the answer is

$$\{x : x \leq -4\} \cup \{x : x \geq 8\}.$$

Exercise 1

Evaluate:

a) $|-11|$ b) $|-9+4|$ c) $-|4|-|-5|$ d) $|-12|-|3|$ e) $|-30| \div |5|$

Answers

a) 11 b) 5 c) -9 d) 9 e) 6

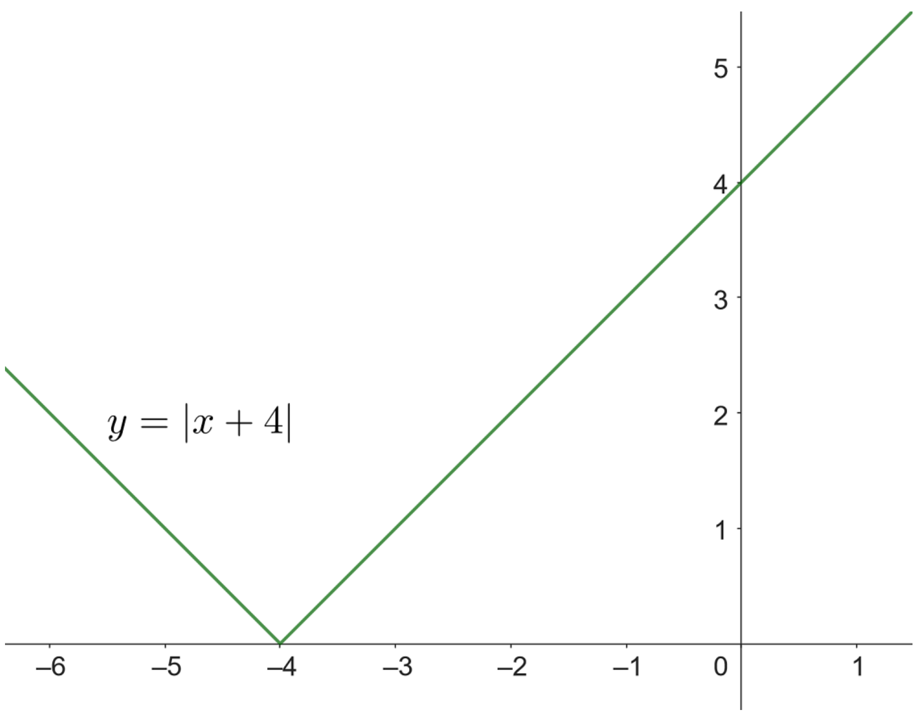
Exercise 2

Sketch the graph of

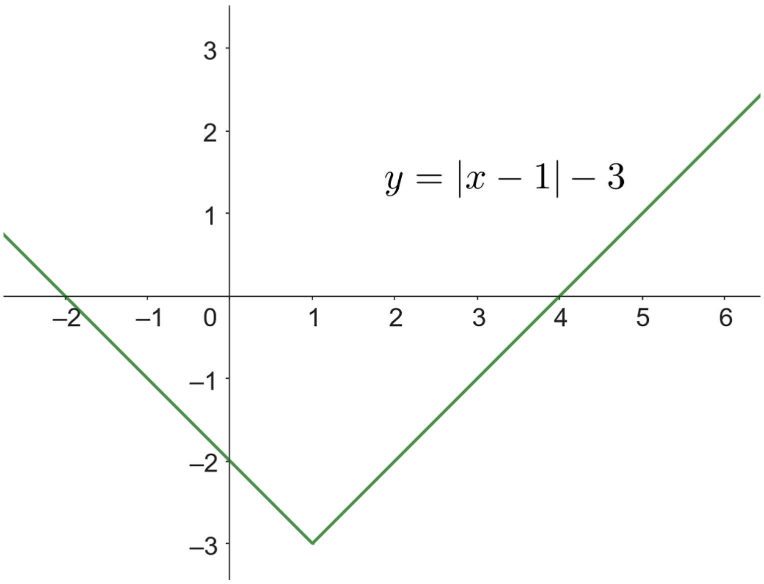
a) $y = |x+4|$ b) $y = |x-1| - 3$ c) $y = |3-x^2|$

Answers

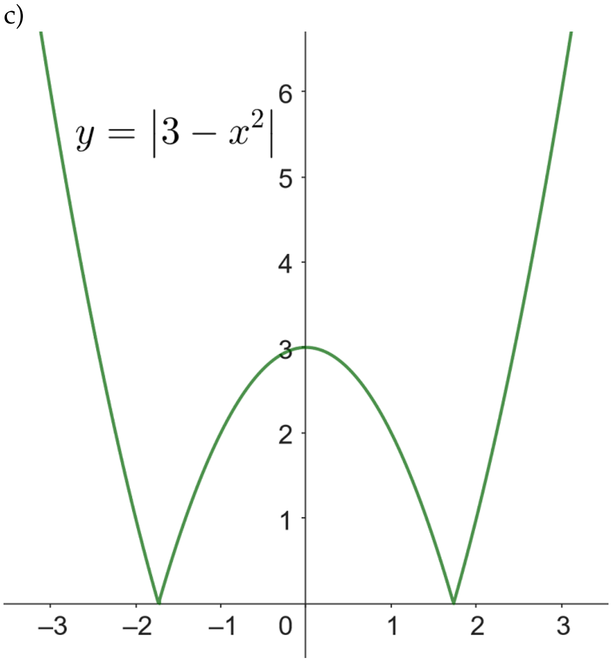
a)



b)



c)



Exercise 3

Find for $x \in \mathbb{R}$

a) $\{x : |x| = 6\}$
b) $\{x : |x-1| < 3\}$
c) $\left\{x : \left|\frac{x-3}{2}\right| \geq 1\right\}$
d) $\left\{x : \left|\frac{x}{2}\right| = |x+2|\right\}$

Answers

a. $\{-6, 6\}$ b. $\{x : -2 < x < 4\}$ c. $\{x : x \leq 1\} \cup \{x : x \geq 5\}$ d. $\left\{-4, -\frac{4}{3}\right\}$