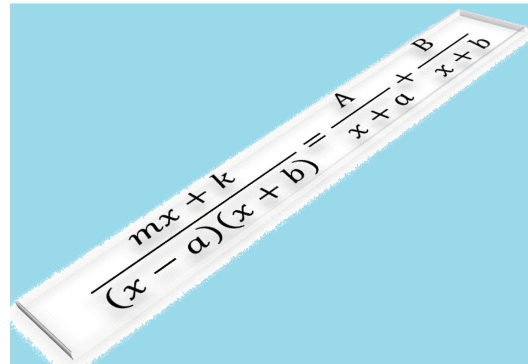
A1.7 Partial Fractions



The method of partial fractions involves breaking up an algebraic fraction into simpler parts that are added together. It is useful in integration of algebraic fractions and also in finding the inverse of Laplace or Fourier transforms.

This module discusses how you write an algebraic fraction in terms of partial fractions.

Adding Fractions

To add fractions, we rewrite the fractions with common denominator and then add the numerators.

For example:

$$\begin{aligned}
\frac{3}{3x+4} + \frac{2}{x-5} &= \frac{3}{(3x+4)} \times \frac{(x-5)}{(x-5)} + \frac{2}{(x-5)} \times \frac{(3x+4)}{(3x+4)} \\
&= \frac{3(x-5)}{(3x+4)(x-5)} + \frac{2(3x+4)}{(3x+4)(x-5)} \\
&= \frac{3(x-5)+2(3x+4)}{(3x+4)(x-5)} \\
&= \frac{3x-15+6x+8}{(3x+4)(x-5)} \\
&= \frac{9x-7}{(3x+4)(x-5)}.
\end{aligned}$$

Finding Partial Fractions

The reverse of this process is to split a fraction into partial fractions.

In the above example,[1]

$$\frac{9x-7}{(3x+4)(x-5)} = \frac{3}{3x+4} + \frac{2}{x-5}.$$

The first step in finding partial fractions is to factorise the denominator. The factorisation will determine the form of the partial

[1] The partial fractions are

$$\frac{3}{3x+4}$$

and

$$\frac{2}{x-5}.$$

fractions:

1. Distinct linear factors $\dfrac{mx+k}{(x+a)(x+b)} = \dfrac{A}{x+a} + \dfrac{B}{x+b}$

2. Repeated linear factors $\dfrac{mx+k}{(x+a)^2} = \dfrac{A}{x+a} + \dfrac{B}{(x+a)^2}$

3. Quadratic and linear factor $\dfrac{mx+k}{(ax^2+bx+c)(px+q)} = \dfrac{Ax+B}{ax^2+bx+c} + \dfrac{C}{px+q}$.

The last step is to determine the constants A, B and C. This is discussed in a later section.

Examples of Partial Fractions

$$\begin{aligned}\frac{2x+1}{x^2+3x+2} &= \frac{2x+1}{(x+2)(x+1)}\\ &= \frac{A}{x+2} + \frac{B}{x+1}\end{aligned}$$

$$\begin{aligned}\frac{x-7}{(2x-1)(x^2+4x+4)} &= \frac{x-7}{(2x-1)(x+2)^2}\\ &= \frac{A}{2x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\end{aligned}$$

$$\frac{x-7}{(2x-1)(x^2+4x+5)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+4x+5}$$

Note that x^2+4x+5 cannot be factorised into two linear factors.

Important Point

The process of finding partial fractions can only be performed on fractions where the degree of the numerator of the algebraic fraction is less than the degree of the denominator (that is, the greatest power of "x" on the top of the fraction must be less than the greatest power of "x" on the bottom of the fraction). If necessary, divide the denominator into the numerator and then express the remaining fractional part as partial fractions.

For example:

$$\begin{aligned}\frac{x^2+7x+7}{x^2+3x+2} &= 1 + \frac{4x+5}{x^2+3x+2}\\ &= 1 + \frac{4x+5}{(x+2)(x+1)}\\ &= 1 + \frac{3}{x+2} + \frac{1}{x+1} \quad \text{(After calculating partial fractions)}\end{aligned}$$

Exercise 1

Rewrite each of the following in the appropriate generalised partial fraction form (do not calculate the constants).

a) $\dfrac{x+6}{2x^2+5x-12}$ b) $\dfrac{2x}{(x^2+3)(x+1)}$

c) $\dfrac{2x}{x^2+8x+16}$ d) $\dfrac{x^2}{x^2-1}$.[2]

[2] Hint: If the power of x in the numerator is greater than or equal to that of the denominator, first perform a long division and then factorise the resulting algebraic fraction.

Answers

a) $\dfrac{A}{2x-3}+\dfrac{B}{x+4}$ b) $\dfrac{Ax+B}{x^2+3}+\dfrac{C}{x+1}$

c) $\dfrac{A}{x+4}+\dfrac{B}{(x+4)^2}$ d) $1+\dfrac{A}{x-1}+\dfrac{B}{x+1}$

Determining the Constants

The constants A, B and C are determined using simultaneous equations or substitution of particular values of x. Both these methods are explained in the following examples.

Example 1: Simultaneous Equations

Express $\dfrac{x-5}{x^2+2x-3}$ as a sum of partial fractions.

$$= \frac{x-5}{(x-1)(x+3)}$$

This is a case of distinct linear factors so

$$\begin{aligned} \frac{x-5}{x^2+2x-3} &= \frac{x-5}{(x-1)(x+3)} \\ &= \frac{A}{x-1}+\frac{B}{x+3} \qquad (1.1) \\ &= \frac{A(x+3)}{(x-1)(x+3)}+\frac{B(x-1)}{(x-1)(x+3)} \\ &= \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}. \qquad (1.2) \end{aligned}$$

From equation (1.1) and (1.2),

$$\begin{aligned} x-5 &= A(x+3)+B(x-1) \qquad (1.3) \\ &= Ax+3A+Bx-B \\ &= (A+B)x+(3A-B). \end{aligned}$$

Equating terms involving x and constants on the LHS and RHS gives:

$$x = (A+B)x$$
$$1 = A + B \text{ assuming } x \neq 0, \qquad (1.4)$$

and

$$-5 = 3A - B\,. \qquad (1.5)\,.$$

Equations (1.4) and (1.5) may be solved simultaneously.[3]
Adding equations (1.4) and (1.5),

[3] There are several way of doing this. The method we use is not unique. However any valid method will lead to the same solution for A and B.

$$-4 = 4A$$
$$4A = -4$$
$$A = -1.$$

Substituting this result in equation (1.4),

$$1 = -1 + B$$
$$B = 2.$$

Using equations (1.1) with $A = -1$ and $B = 2$ gives the solution:

$$\frac{x-5}{x^2+2x-3} = \frac{-1}{x-1} + \frac{2}{x+3}.$$

Example 1: Alternative method.

In the solution above we reached this step (see equation (1.3)):

$$x - 5 = A(x+3) + B(x-1).$$

Instead of equating coefficients and solving simultaneous equations, we can often solve for A and B by substituting any convenient value of x into this equation.

For example, if we let $x = 1$ we can eliminate B from the equation and solve for A.

If $x = 1$

$$1 - 5 = A(1+3) + B(1-1)$$
$$-4 = A(4) + B(0)$$
$$-4 = 4A$$
$$A = -1$$

If we let $x = -3$ we can eliminate A and solve for B.

If $x = -3$

$$\begin{aligned}
-3-5 &= A(-3+3) + B(-3-1) \\
-8 &= A(0) + B(-4) \\
-8 &= -4B \\
B &= 2
\end{aligned}$$

Therefore

$$\frac{x-5}{(x-1)(x+3)} = \frac{-1}{x-1} + \frac{2}{x+3}$$

This method, of course, gives the same answer as when you equate coefficients and solve simultaneous equations.

Example 2

Express $\dfrac{5x^2+3x+1}{x^3-3x-2}$ as the sum of partial fractions.

$$\frac{5x^2+3x+1}{x^3-3x-2} = \frac{5x^2+3x+1}{(x+1)^2(x-2)} \quad \text{[factorise the denominator]}$$

$$\frac{5x^2+3x+1}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} \quad \text{[repeated linear factors]}$$

$$\frac{5x^2+3x+1}{(x+1)^2(x-2)} = \frac{A(x+1)(x-2) + B(x-2) + C(x+1)^2}{(x+1)^2(x-2)} \quad \text{[finding a common denominator]}$$

$$5x^2+3x+1 = A(x+1)(x-2) + B(x-2) + C(x+1)^2 \quad \text{[equating numerators]}$$

We can now substitute any convenient value for x. [4]
Let $x = 2$

[4] Here we are using the alternate method described in Example 1 above.

$$\begin{aligned}
5x^2+3x+1 &= A(x+1)(x-2) + B(x-2) + C(x+1)^2 \\
&\text{substitute } x = 2 \\
5(2)^2+3(2)+1 &= A(2+1)(2-2) + B(2-2) + C(2+1)^2 \\
27 &= A(0) + B(0) + 9C \\
27 &= 9C \\
C &= 3.
\end{aligned}$$

Let $x = -1$

$$\begin{aligned}
5x^2+3x+1 &= A(x+1)(x-2) + B(x-2) + C(x+1)^2 \\
5(-1)^2+3(-1)+1 &= A(-1+1)(-1-2) + B(-1-2) + C(-1+1)^2 \\
3 &= A(0) + B(-3) + C(0) \\
3 &= -3B \\
B &= -1.
\end{aligned}$$

Substitute in the values of C and B and let $x = 0$

$$\begin{aligned}
5x^2 + 3x + 1 &= A(x+1)(x-2) + B(x-2) + C(x+1)^2 \\
5(0)^2 + 3(0) + 1 &= A(0+1)(0-2) + (-1)(0-2) + (3)(0+1)^2 \\
1 &= -2A + 2 + 3 \\
1 - 2 - 3 &= -2A \\
-4 &= -2A \\
A &= 2.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{5x^2+3x+1}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2} \\
\implies &\frac{5x^2+3x+1}{(x+1)^2(x-2)} = \frac{2}{x+1} - \frac{1}{(x+1)^2} + \frac{3}{x-2}.
\end{aligned}$$

Example 3

Express $\dfrac{x^2-3x+16}{x^3-5x^2+x-5}$ as a sum of partial fractions.

We have

$$\begin{aligned}
\frac{x^2-3x+16}{x^3-5x^2+x-5} &= \frac{x^2-3x+16}{x^2(x-5)+(x-5)} \\
&= \frac{x^2-3x+16}{(x^2+1)(x-5)} \\
&= \frac{Ax+B}{x^2+1} + \frac{C}{x-5} \qquad (3.1) \\
&= \frac{(Ax+B)(x-5)}{(x^2+1)(x-5)} + \frac{C(x^2+1)}{(x-5)(x^2+1)}
\end{aligned}$$

and so

$$\begin{aligned}
x^2 - 3x + 16 &= (Ax+B)(x-5) + C(x^2+1) \\
&= Ax^2 - 5Ax + Bx - 5B + Cx^2 + C \\
&= (A+C)x^2 + (-5A+B)x + (-5B+C) \qquad (3.2)
\end{aligned}$$

Equating terms on the LHS and RHS we have

$$\begin{aligned}
\text{for } x^2 \quad & 1 = A + C & (3.3) \\
\text{for } x \quad & -3 = -5A + B & (3.4) \\
\text{for } x^0 = \text{constant} \quad & 16 = -5B + C & (3.5)
\end{aligned}$$

We need to solve these simultaneously.[5] Multiply equation (3.3)

[5] As mentioned previously, there are several approaches we can use. Some may be more efficient than others but the result will be the same.

by 5 and add it to equation (3.4)

$$\begin{aligned} 5 - 3 &= 5A + 5C + -5A + B \\ 2 &= 5C + B \\ B &= 2 - 5C. \end{aligned}$$

Substitute this in equation (3.5):

$$\begin{aligned} 16 &= -5\,(2 - 5C) + C \\ &= -10 + 25C + C \\ &= -10 + 26C \\ 26C &= 26 \\ C &= 1. \end{aligned}$$

Substitute $C = 1$ in equation (3.3)

$$\begin{aligned} 1 &= A + 1 \\ A &= 0. \end{aligned}$$

Substitute $A = 0$ in equation (3.4)

$$\begin{aligned} -3 &= B \\ B &= -3. \end{aligned}$$

Finally, we get our solution by substituting $A = 0$, $B = -3$ and $C = 1$ into equation (3.1)

$$\frac{x^2 - 3x + 16}{x^3 - 5x^2 + x - 5} = \frac{-3}{x^2 + 1} + \frac{1}{x - 5}.$$

Summary

To express an algebraic fraction as partial fractions:

1. Factorise the denominator.
2. Write the algebraic fraction as the sum of partial fractions with unknown constants as above.
3. Add the partial fractions by getting a common denominator.
4. Then equate the coefficients and use simultaneous equations, or substitute a value of x to determine the values of the constants.

Exercise 2

Express the following as partial fractions.

a) $\dfrac{x+2}{x^2 - 5x + 6}$ b) $\dfrac{3}{x^2 - 2x + 1}$

c) $\dfrac{x^2 - 2x + 2}{x^3 + x^2 + x}$ d) $\dfrac{x^2}{x^2 - 4}$.

Answers

a) $\dfrac{5}{x-3} - \dfrac{4}{x-2}$ b) $\dfrac{3}{(x-1)^2}$

c) $\dfrac{2}{x} - \dfrac{x+4}{x^2+x+1}$ d) $1 - \dfrac{1}{(x+2)} + \dfrac{1}{(x-2)}$.