A1.6 Algebraic Fractions (Quadratics)

$$\frac{\dfrac{-3}{x^2+2x-3}+\dfrac{1}{x-1}}{\dfrac{3}{x-1}+3}$$

More complicated algebraic fractions can involve polynomials of any order.

However in your studies, it is unlikely that you will have to deal with polynomials higher than quadratics. [1]

In this module, we consider algebraic fractions involving quadratics.

There is no general theory for this so we proceed via examples. Please note that algebraic fraction may exclude certain values of x. This is to ensure we never get a zero denominator. [2]

The concept of a common denominator is fundamental to algebraic fractions.[3]

Example 1.

Determine

$$\frac{2}{x^2+x-12}-\frac{1}{x^2-9}.$$

Solution:

You could use $\left(x^2+x-12\right)\times\left(x^2-9\right)$ as a common denominator and work from there. However, that would give you a quartic[4] as a denominator which may be a bit difficult to reduce to lowest terms. Instead we note that the denominators may be factorised. In particular,

$$x^2+x-12=(x+4)(x-3)$$

and

$$x^2-9=(x-3)(x+3).$$

So we can write

$$\frac{2}{x^2+x-12}-\frac{1}{x^2-9}=\frac{2}{(x+4)(x-3)}-\frac{1}{(x-3)(x+3)}.$$

A common denominator is

$$(x+4)(x-3)(x+3)$$

[1] A quadratic is a a polynomial of degree 2 and has the general form:

$$ax^2+bx+c$$

where a,b,c are constants.

[2] The numerator is the top expression in the fraction. For

$$\frac{a}{b}$$

a is the numerator. The bottom expression is called the denominator. Here b is the denominator.

[3] Before you can add or subtract two fractions you need to have a common denominator. For example, to determine

$$\frac{2}{3}-\frac{1}{2}$$

we have to convert each fraction into sixths as a common multiple of 3 and 2 is 6. Therefore

$$\begin{aligned}\frac{2}{3}-\frac{1}{2}&=\frac{4}{6}-\frac{3}{6}\\&=\frac{4-3}{6}\\&=\frac{1}{6}.\end{aligned}$$

The same concept is necessary for algebraic fractions.

[4] A quartic is a polynomial with highest term of degree 4. That is a term involving x^4.

so

$$\begin{aligned}
\frac{2}{x^2+x-12}-\frac{1}{x^2-9} &= \frac{2\left(x+3\right)}{\left(x+4\right)\left(x-3\right)\left(x+3\right)}-\frac{1\left(x+4\right)}{\left(x-3\right)\left(x+3\right)\left(x+4\right)} \\
&= \frac{2\left(x+3\right)-1\left(x+4\right)}{\left(x+4\right)\left(x-3\right)\left(x+3\right)} \\
&= \frac{2x+6-x-4}{\left(x+4\right)\left(x-3\right)\left(x+3\right)} \\
&= \frac{x+2}{\left(x+4\right)\left(x-3\right)\left(x+3\right)} \text{ where } x \neq -4,-3,3.
\end{aligned}$$

Example 2

Simplify

$$\frac{1-\frac{2}{x}}{1+\frac{2}{x}}.$$

Solution:

$$\begin{aligned}
\frac{1-\frac{2}{x}}{1+\frac{2}{x}} &= \left(1-\frac{2}{x}\right)\div\left(1+\frac{2}{x}\right) \\
&= \frac{x-2}{x}\div\frac{x+2}{x} \\
&= \frac{x-2}{x}\times\frac{x}{x+2} \\
&= \frac{x-2}{x+2} \text{ where } x \neq -2.
\end{aligned}$$

Example 3

Simplify

$$\frac{\frac{-3}{x^2+2x-3}+\frac{1}{x-1}}{\frac{3}{x-1}+3}. \qquad (3.1)$$

Solution:

The numerator is:

$$\begin{aligned}
\frac{-3}{x^2+2x-3}+\frac{1}{x-1} &= \frac{-3}{\left(x+3\right)\left(x-1\right)}+\frac{1}{x-1} \\
&= \frac{-3}{\left(x+3\right)\left(x-1\right)}+\frac{1}{x-1}.
\end{aligned}$$

Using a common denominator, we have

$$\begin{aligned}\frac{-3}{x^2+2x-3}+\frac{1}{x-1} &= \frac{-3}{(x+3)(x-1)}+\frac{1}{x-1}\\ &= \frac{-3}{(x+3)(x-1)}+\frac{(x+3)}{(x+3)(x-1)}\\ &= \frac{x}{(x+3)(x-1)}. \qquad (3.2)\end{aligned}$$

Now consider the denominator of eqn(3.1):

$$\begin{aligned}\frac{3}{x-1}+3 &= \frac{3}{x-1}+\frac{3(x-1)}{x-1}\\ &= \frac{3x}{x-1}. \qquad (3.3)\end{aligned}$$

Using eqns (3.2) and (3.3)

$$\begin{aligned}\frac{\frac{-3}{x^2+2x-3}+\frac{1}{x-1}}{\frac{3}{x-1}+3} &= \frac{x}{(x+3)(x-1)} \div \frac{3x}{x-1}\\ &= \frac{x}{(x+3)(x-1)} \times \frac{(x-1)}{3x}\\ &= \frac{x}{3x(x+3)}\\ &= \frac{1}{3(x+3)} \text{ where } x \neq -3.\end{aligned}$$

Exercises

Simplify the following:

1.

$$\frac{2}{x^2+2x}+\frac{1}{x^2-4}$$

2.

$$\frac{1-\frac{6}{x}}{\frac{x}{2}-3}$$

3.

$$\frac{\frac{1}{x^2-4}+\frac{1}{2x+4}}{1+\frac{2}{x-2}}$$

Answers

1.

$$\frac{3x-4}{x(x+2)(x-2)} \text{ where } x \neq -2, 0, 2.$$

2.

$$\frac{2}{x} \text{ where } x \neq 0.$$

3.

$$\frac{1}{2\,(x+2)} \text{ where } x \neq -2.$$