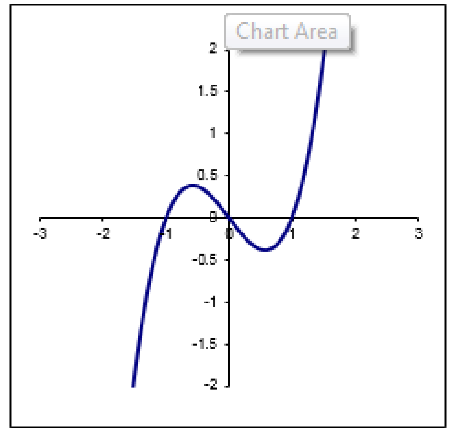
D9: Curve Sketching



To sketch a curve it is helpful to find the

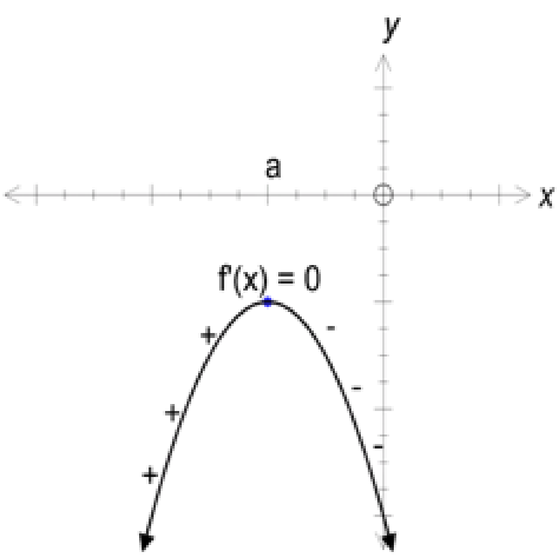
1. x and y intercepts
2. Maximum and minimum points.

This module describes how to do this.

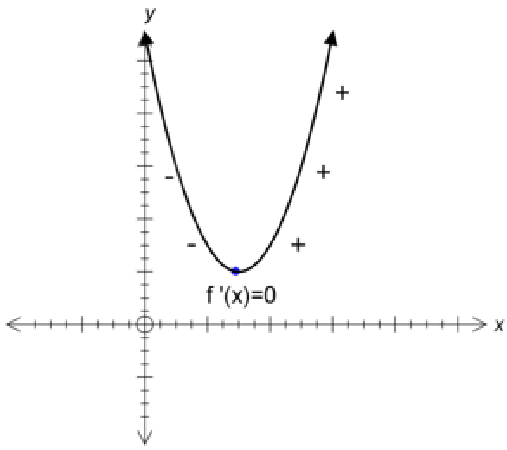
Definition of Maximum and Minimum

A stationary point is a point on a graph of a function $y = f(x)$ where the tangent to the curve is horizontal. At a stationary point the derivative function $y = f'(x) = 0$.

A **maximum** stationary point occurs at $x = a$ if $f'(a) = 0$ and $f'(x) > 0$ for $x < a$ and $f'(x) < 0$ for $x > a$ as shown below.



A **minimum** stationary point occurs at $x = a$ if $f'(a) = 0$ and $f'(x) < 0$ for $x < a$ and $f'(x) > 0$ for $x > a$ as shown below.



Example 1

Find the turning point of the parabola defined by $y = x^2 + 4x + 5$ and determine if it is a maximum or minimum.

Solution

Let

$$f(x) = x^2 + 4x + 5$$

then

$$f'(x) = 2x + 4.$$

At a stationary point, $f'(x) = 0$, so

$$\begin{aligned} 2x + 4 &= 0 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

is the $x-$coordinate of a stationary point. When $x = -2$,

$$\begin{aligned} f(-2) &= (-2)^2 + 4(-2) + 5 \\ &= 1. \end{aligned}$$

Hence the coordinates of the stationary point are $(-2, 1)$. Now we ascertain if it is a maximum or minimum.

A sign test can be used to determine whether the stationary point is minimum or a maximum by checking the slope of the tangent on each side of the stationary point. Consider the table below.

x	-2.1	-2	-1.9
$f'(x)$	- ve	0	+ ve
gradient	\	-	/

As we move from the left to the right of the stationary point at $x = -2$, the gradient changes from negative to positive. This indicates there is a minimum at $(-2, 1)$.

Hence the turning point of the parabola is a minimum and occurs at $(-2, 1)$.

Example 2

Sketch the graph of $y = x^3 - x$.

Solution strategy:

1. Find intercepts on $x-$axis.
2. Find stationary points.
3. Establish if stationary points are maximum or minimum values.
4. Plot intercepts and stationary points and sketch the graph.

Solution

For $x-$axis intercepts, we set $y = 0$, so

$$\begin{aligned} 0 &= x^3 - x \\ &= x\left(x^2 - 1\right). \end{aligned}$$

Consequently, $x = 0$ or

$$\begin{aligned} x^2 - 1 &= 0 \\ x &= 1 \text{ or } -1. \end{aligned}$$

Hence the $x-$axis intercepts are $(-1, 0)$, $(0, 0)$ and $(1, 0)$.

For stationary points,

$$\frac{dy}{dx} = 0$$

that is

$$\begin{aligned} 3x^2 - 1 &= 0 \\ x^2 &= \frac{1}{3} \\ x &= \pm\frac{1}{\sqrt{3}} \\ &\approx \pm 0.58. \end{aligned}$$

Substituting these x values back in $y = x - x^3$ gives for $x = 1/\sqrt{3}$,

$$y = \left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}}$$
$$\approx -0.39.$$

For $x = -1/\sqrt{3}$

$$y = \left(-\frac{1}{\sqrt{3}}\right)^3 - \left(-\frac{1}{\sqrt{3}}\right)$$
$$\approx 0.38.$$

Hence the stationary points occur at approximately, $(0.58, -0.39)$ and $(-0.58, 0.38)$. Now we decide if these points are maxima or minima. Consider the tables below.

For $x \approx 0.58$ we have

x	0.5	0.58	0.6
$f'(x)$	- ve	0	+ ve
gradient	\	-	/

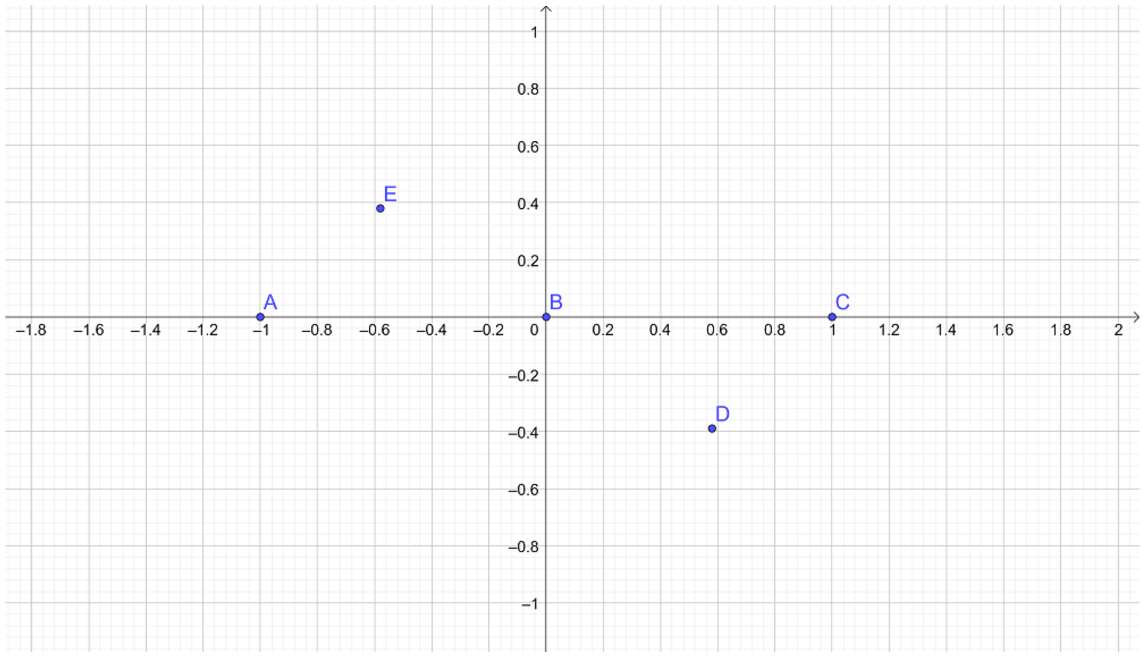
and so the point $x = 1/\sqrt{3} \approx 0.58$ is a minimum.

For $x \approx -0.58$ we have

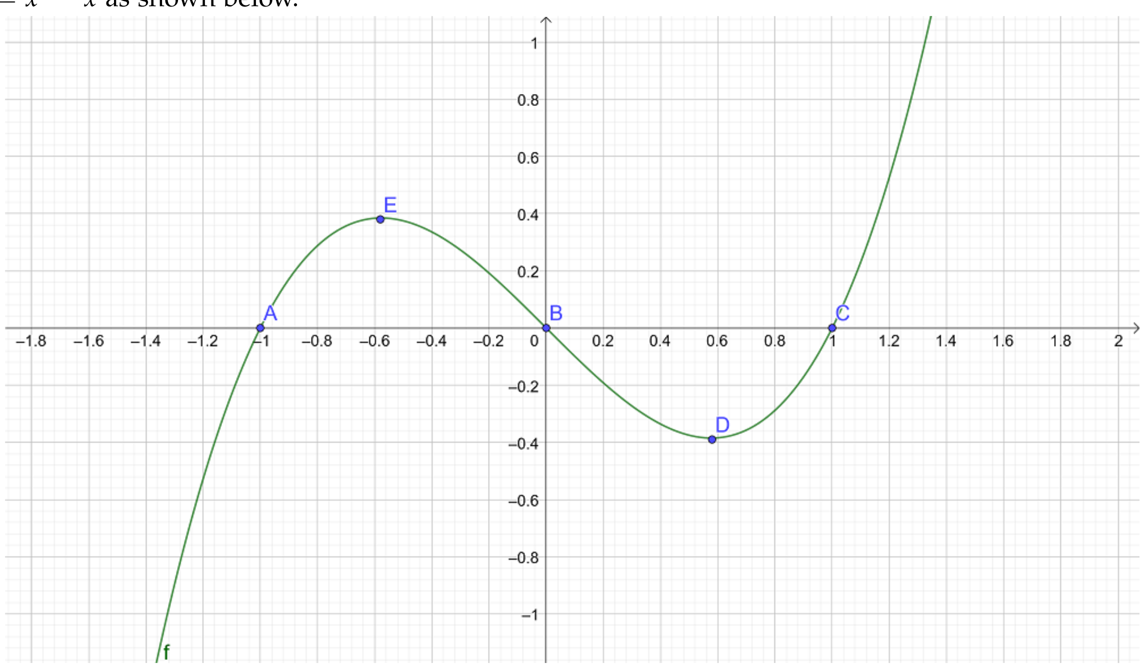
x	-0.6	-0.58	-0.5
$f'(x)$	+ ve	0	- ve
gradient	/	-	\

and so the point $x = -1/\sqrt{3} \approx -0.58$ is a maximum.

We can now graph $y = x^3 - x$. First plot the $x-$intercepts and the stationary points:



We know E is a maximum and D is a minimum and so can graph $y = x^3 - x$ as shown below.



Exercise

Sketch the graphs of the following functions showing all intercepts and turning points

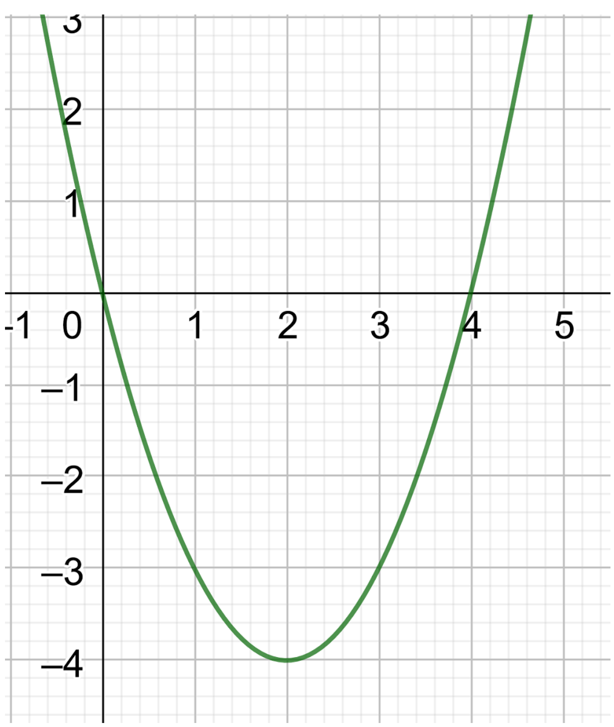
1. $y = x^2 - 4x$

2. $y = x^3 - 2x^2 + x$

3. $y = 6 - x - x^2$

4. $y = (x+1)^4$

Answers

1. 

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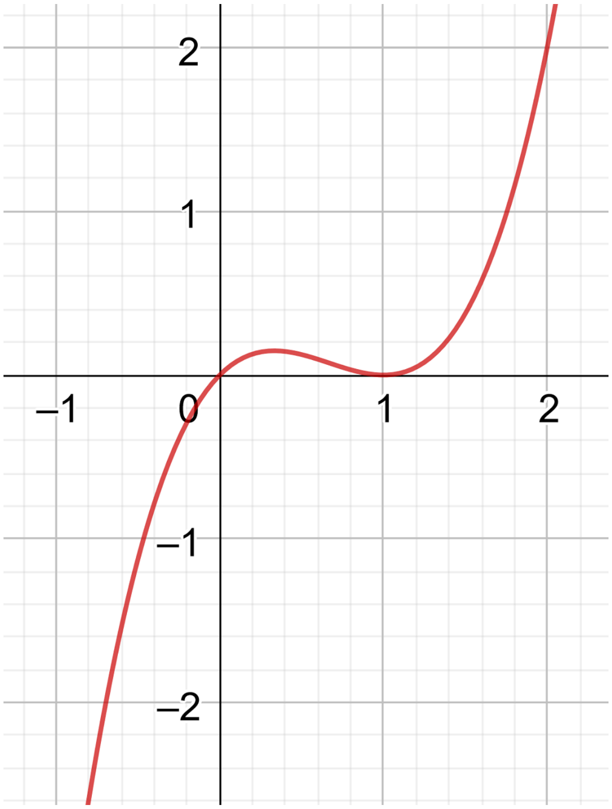

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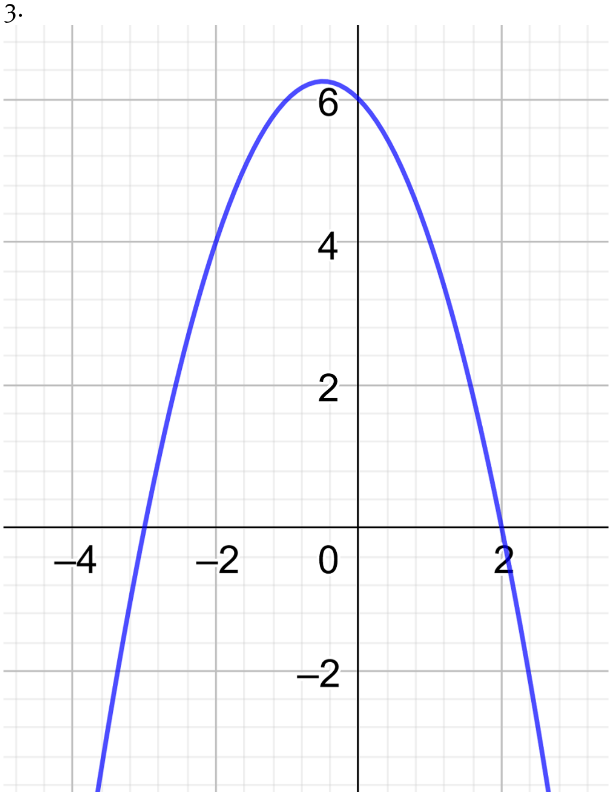

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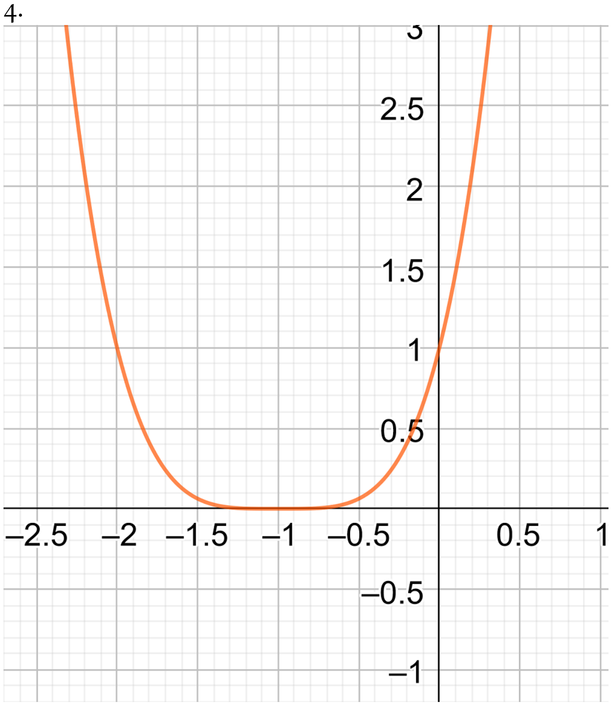

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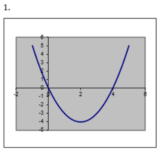

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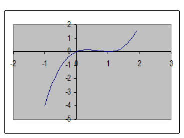

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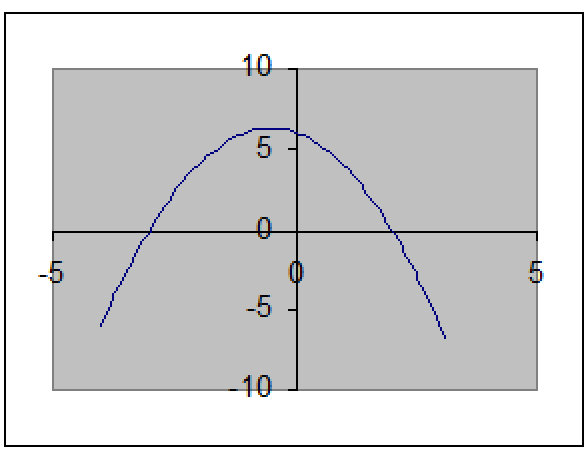

4.