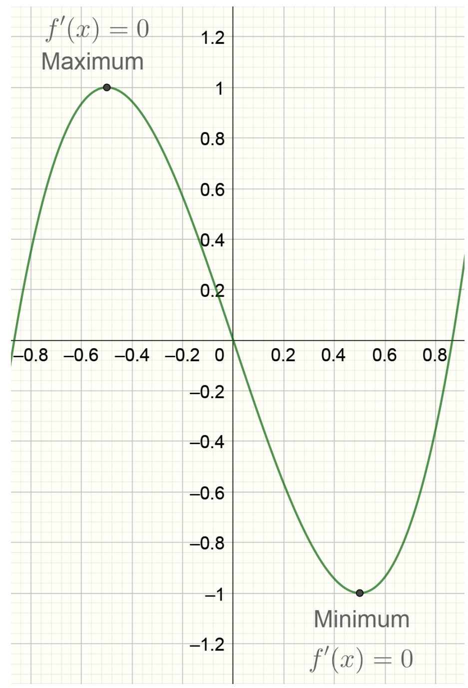
D8: Maxima and Minima



The maximum or minimum values of a function occur where the derivative is zero. That is where the graph of teh function has a horizontal tangent. Using calculus we can find trhe derivative of a function $f(x)$ with respect to x and use this to find maximum and minimum values of $f(x)$ and the values of x where they occur. We can therefore use calculus to solve problems that involve maximizing or minimizing functions.

Definition

The maximum or minimum values of a finction $f(x)$ occur when the derivative

$$f'(x) = 0. \tag{1}$$

Second Derivative Test.

Let x satisfy (1) then if

$$f''(x) \begin{cases} = 0, & x \text{ is an inflection point} \\ < 0, & f(x) \text{ is a local maximum} \\ > 0, & f(x) \text{ is a local minimum.} \end{cases}$$

Example 1

The distance $s\,km$, to the nearest km, of a fishing boat from port at any time, t hours, is given by the formula

$$s = 2 + 8t – 2.5t^2.$$

When is the boat furthest from port and what is its distance from the port at that time?

Solution

For a maximum or minimum, $ds/dt = 0$. That is

$$\frac{ds}{dt} = 8 - 5t$$
$$= 0.$$

So,

$$8 - 5t = 0$$
$$5t = 8$$
$$t = 1.6 \text{ hours.}$$

Now this could be a maximum or minimum distance. However,

$$\frac{d^2s}{dt^2} = -5$$
$$< 0.$$

Hence $t = 1.6$ is a maximum. When $t = 1.6$ hours, the distance from port,

$$s = 2 + 8(1.6)–2.5(1.6)^2$$
$$= 8.4\,kms.$$

The boat is furthest from port after 1.6 hours and the distance from port, at that time, is $8.4\,kms$.

Example 2

Find the maximum product of two numbers that have a sum of 10.

Solution

Let the numbers be a and b. Then

$$a + b = 10. \tag{2.1}$$

Let the product of the two numbers be P so

$$P = a \times b.$$

From (2.1)[1]

$$a = 10 - b$$

so

$$P = (10 - b)\,b$$
$$= 10b - b^2.$$

Now

$$\frac{dP}{db} = 10 - 2b.$$

[1] We need to get P in terms of a or b so that we take a derivative like dP/da or dP/db. In this case we write P as a function of b but identical results are obtained is we make P a function of a.

For a maximum or minimum, $dP/db = 0$,

$$\begin{aligned} 10 - 2b &= 0 \\ 2b &= 10 \\ b &= 5. \end{aligned} \tag{2.2}$$

But

$$\begin{aligned} \frac{d^2P}{db^2} &= -2 \\ &< 0 \end{aligned}$$

and we have a maximum. Substituting $b = 5$ in (2.1) we find $a = 5$.

Hence the two numbers adding to 10 and having a maximal product are $a = b = 5$ and the maximum product is 25.

Example 3

Find the minimum value of the function $f(x) = x^2 - 5x + 6$.

Solution

We have

$$f'(x) = 2x - 5. \tag{3.1}$$

For a maximum or minimum we know

$$\begin{aligned} f'(x) &= 0 \\ 2x - 5 &= 0 \\ 2x &= 5 \\ x &= \frac{5}{2}. \end{aligned}$$

Since

$$\begin{aligned} f''(x) &= 2 \\ &> 0 \end{aligned}$$

for all values of x we know we have a minimum. Hence the minimum value of $f(x) = x^2 - 5x + 6$ occurs at $5/2$.

The minimum value of the function is

$$\begin{aligned} f(x) &= \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 6 \\ &= \frac{25}{4} - \frac{25}{2} + \frac{12}{2} \\ &= \frac{25}{4} - \frac{50}{4} + \frac{24}{4} \\ &= -\frac{1}{4}. \end{aligned}$$

Exercises

1. Find two positive numbers whose sum is 18 such that the sum of their squares is a minimum.
2. Find the turning point of the parabola defined by $y = f(x) = 5x^2 - 30x + 17$.
3. What is the maximum area that can be enclosed if a rectangle is created with a piece of wire $48\,cm$ long?
4. The annual profit P made on a garment is related to the number n that are produced by the formula $P(n) = 300n - 7200 - 0.2n^2$. How many garments should be produced to maximize profit?

Answers

1. The two numbers are both 9.
2. $(3, -28)$
3. 144cm^2
4. 750