RMIT UNIVERSITY

D7: The Quotient Rule

$$y = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v(x)\frac{d}{dx}u(x) - \frac{d}{dx}v(x)u(x)}{(v(x))^2}$$

$$y' = \frac{vu' - v'u}{v^2}$$

The quotient rule is used when we want to differentiate a function which is the quotient of two simpler functions. Functions such as $y = f(x) = \frac{1}{x^2+x}$, $y = f(x) = \frac{\sin x}{x}$ and $y = f(x) = \frac{x^2+1}{x+1}$ may be differentiated using the quotient rule.

Definition

If

$$f(x) = \frac{u(x)}{v(x)}$$

then

$$f'(x) = \frac{v(x)\,u'((x)) - u(x)\,v'(x)}{(v(x))^2}.$$

This is often abbreviated to

$$\begin{aligned} y' &= f'(x) \\ &= \frac{vu' - uv'}{v^2} \end{aligned}$$

View short video on the quotient rule.

Examples

1) If $y = \frac{1+x}{x^2-3}$, find $\frac{dy}{dx}$.

Solution

Let

$$u = 1 + x \text{ and } v = x^2 - 3$$

then

$$u' = 1 \text{ and } v' = 2x.$$

Hence using the quotient rule,

$$\begin{aligned}
\frac{dy}{dx} &= y' \\
&= \frac{vu' - uv'}{v^2} \\
&= \frac{\left(x^2-3\right)(1) - (1+x)\,2x}{\left(x^2-3\right)^2} \\
&= \frac{x^2 - 3 - 2x - 2x^2}{(x^2-3)^2} \\
&= \frac{-x^2 - 2x - 3}{(x^2-3)^2}.
\end{aligned}$$

2) Differentiate $\frac{x^2}{\log_e x}$with respect to x.
Solution
Let

$$y = \frac{x^2}{\log_e (x)}$$

and

$$\begin{aligned}
u &= x^2 \\
v &= \log_e (x)\,.
\end{aligned}$$

Then

$$u' = 2x \text{ and } v' = \frac{1}{x}.$$

Hence, using the quotient rule,

$$\begin{aligned}
y' &= \frac{vu' - uv'}{v^2} \\
&= \frac{\log_e (x) \cdot (2x) - x^2 \cdot \frac{1}{x}}{(\log_e x)^2} \\
&= \frac{2x \log_e (x) - x}{(\log (x))^2}.
\end{aligned}$$

Exercise

Find the derivatives of the following functions with respect to x.

1) $f(x) = \frac{2x+1}{4x-3}$
2) $f(x) = \frac{3}{3x^2+1}$
3) $y = \frac{\sqrt{x}}{1-\sqrt{x}}$
4) $y = \frac{e^x}{\sin^2 x}$

Answers

1) $f'(x) = \frac{-10}{(4x-3)^2}$

2) $f'(x) = \frac{-18x}{(3x^2+1)^2}$

3) $y' = \frac{1}{2x^{\frac{1}{2}}\left(1-x^{\frac{1}{2}}\right)^2}$ (after simplifying)

4) $y' = \frac{e^x(\sin x - 2\cos x)}{\sin^3 x}$ (after simplifying)