D6: The Product Rule

$$y = v(x)u(x)$$

$$\frac{dy}{dx} = v(x)\frac{d}{dx}u(x) + \frac{d}{dx}v(x)\,u(x)$$

$$y' = vu' + v'u$$

The product rule is used when we want to differentiate the product of two functions. The derivatives of functions such as $y = f(x) = 2x\sin(x)$ and $y = f(x) = xe^x$ can be found using the product rule.

Definition

If

$$\begin{aligned} y &= f(x) \\ &= u(x) \cdot v(x) \end{aligned}$$

then

$$\begin{aligned} y' &= f'(x) \\ &= u(x) \cdot v'(x) + u'(x) \cdot v(x). \end{aligned}$$

This is often abbreviated to

$$y' = uv' + u'v.$$

View short video on the product rule.

Examples

1. Find the derivative of $f(x) = (x+3)^6(2x-1)$.
   Solution:
   Let $u = (x+3)^6$ and $v = 2x - 1$, then using the chain rule[1]

$$u' = 6(x+3)^5$$

and

$$v' = 2.$$

Hence, using the product rule,

[1] Let $w = x + 3$ then $u = w^6$ and

$$\begin{aligned} u' &= \frac{du}{dw} \times \frac{dw}{dx} \\ &= 6w^5 \cdot 1 \\ &= 6(x+3)^5. \end{aligned}$$

$$\begin{aligned} y' &= uv' + u'v \\ &= (x+3)^6 \cdot 2 + 6(x+3)^5 (2x–1) \\ &= (x+3)^5 [2(x+3) + 6(2x-1)] \\ &= 14x(x+3)^5. \end{aligned}$$

2. Differentiate $e^x \sin(2x)$.

Solution:

Let $u = e^x$ and $v = \sin(2x)$, then

$$\begin{aligned} u' &= e^x \\ v' &= 2\cos(2x) \end{aligned}$$

where we have used the chain rule to evaluate v'.[2] Hence, using the product rule,

$$\begin{aligned} y' &= uv' + u'v \\ &= e^x \cdot 2\cos(2x) + e^x \sin(2x) \\ &= 2e^x \cos(2x) + e^x \sin(2x) \\ &= e^x (2\cos(2x) + \sin(2x)). \end{aligned}$$

[2] Let $w = 2x$ then $v = \sin(w)$ and

$$\begin{aligned} v' &= \frac{dv}{dw} \times \frac{dw}{dx} \\ &= \cos(w) \cdot 2 \\ &= 2\cos(2x). \end{aligned}$$

Exercises

1. Use the product rule to differentiate the following
   a) $y = (x-2)(6x+7)$ and simplify as far as possible.
   b) $f(x) = \left(2x^2+4\right)\left(x^5+4x^2-2\right)$ (do not simplify).
   c) $y = \left(\sqrt{x}-1\right)\left(x^2+1\right)$
   d) $y = \left(x^3-4x+\sqrt{x}\right)\left(3x^4+2\right)$.

   Answers (Note that answers may be written differently)
   a) $12x-5$
   b) $\left(2x^2+4\right)\left(5x^4+8x\right)+4x\left(x^5+4x^2-2\right)$
   c) $\frac{5}{2}x^{3/2} - 2x + \frac{1}{2\sqrt{x}}$
   d) $12x^3\left(x^3-4x+\sqrt{x}\right)+\left(3x^4+2\right)\left(3x^2-4+\frac{1}{2\sqrt{x}}\right)$
2. Find the derivative of
   a) $y = e^x \tan x$
   b) $y = x^2 \log_e x$
   c) $y = \sin x \cos x$
   d) $y = \frac{e^x}{x}$ Hint : $\frac{1}{x} = x^{-1}$.

   Answers (Note that answers may be written differently)
   a) $e^x \tan x + e^x \sec^2 x$
   b) $x + 2x\log_e x$
   c) $\cos^2(x) - \sin^2(x)$
   d) $\frac{e^x}{x} - \frac{e^x}{x^2} = e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$