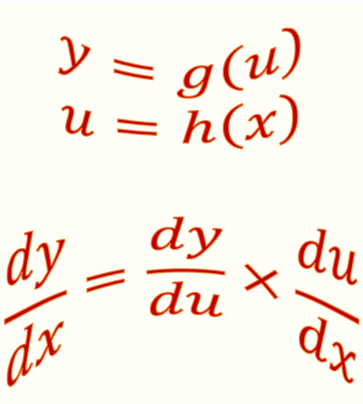
D5 *The Chain Rule*



The "chain rule" is used to differentiate a function which is the composition of two simpler functions. The derivatives of functions such as $y = \sin(x^3)$ and $f(x) = (x^2 - 1)^4$ can be found using the chain rule.

View a short video on the chain rule.

The Chain Rule for Differentiation

If $y = g(u)$ where $u = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Examples

1) Differentiate $y = (2x - 1)^4$

Let u= 2x - 1 $\Rightarrow$ y= u^4

Then $\frac{du}{dx}$ =2 and $\frac{dy}{du} = 4u^3$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 4u^3.2 \\ &= 8u^3 \\ &= 8(2x-1)^3 \qquad \text{[since u = 2x-1]}\end{aligned}$$

2) *Find the derivative* of y= $\frac{1}{\sqrt[3]{5t^2+2t+1}}$

y= $(5t^2 + 2t + 1)^{-\frac{1}{3}}$ [change to index form for differentiation]

Let $u = 5t^2 + 2t + 1 \Rightarrow y = u^{-\frac{1}{3}}$

Then $\frac{du}{dt} = 10t + 2$ and $\frac{dy}{du} = -\frac{1}{3}u^{-\frac{4}{3}}$

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{du} \times \frac{du}{dt} \\ &= \left(-\frac{1}{3}u^{-\frac{4}{3}}\right)(10t+2) \\ &= -\frac{10t+2}{3}\left(5t^2+2t+1\right)^{-\frac{4}{3}} \text{ [after simplifying]}\end{aligned}$$

3) *Differentiate* $y = \sin(5x)$
Let $u = 5x \Rightarrow y = \sin(u)$
Then $\frac{du}{dx} = 5$ and $\frac{dy}{du} = \cos(u)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \cos(u) \times 5 \\ &= 5\cos(5x)\end{aligned}$$

4) *If* $f(x)=\cos^3 x$ *find* $f'(x)$
$y = \cos^3 x = [\cos(x)]^3$
Let $u = \cos(x) \Rightarrow y=u^3$
Then $\frac{du}{dx} = -\sin(x)$ and $\frac{dy}{du} = 3u^2$

$$\begin{aligned}f'(x) = \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 3u^2 \times [-\sin(x)] \\ &= 3\cos^2 x \times [-\sin(x)] \\ &= -3\sin x \cos^2 x\end{aligned}$$

5) *Differentiate* $y = (\log_e [4x])^3$
NB:y is a composite of THREE functions
Let $v = 4x$ and $u = \log_e v \Rightarrow y = u^3$
Then $\frac{dv}{dx} = 4$ and $\frac{du}{dv} = \frac{1}{v}$and $\frac{dy}{du} = 3u^2$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} \\ &= 3u^2 \times \frac{1}{v} \times 4 \\ &= 3(\log_e v)^2 \cdot \frac{1}{4x} \cdot 4 \\ &= \frac{3}{x}(\log_e [4x])^2\end{aligned}$$

Exercise

Find the derivatives of the following functions

1. y= $\tan 3x$
2. $f(x)$=$\log_e \frac{x}{2}$
3. $y = \sin\left(\frac{\pi}{4} - 2x\right)$
4. $y = \cos^2 x$
5. $f(x) = e^{\sin x}$
6. $y = \cos^2(10x)$

Answers

1. $y' = 3\sec^2(3x)$
2. $f'(x) = \frac{1}{x}$
3. $y' = -2\cos\left(\frac{\pi}{4} - 2x\right)$
4. y'=-2$\sin x \cos x$
5. $f'(x) = e^{\sin x}\cos x$
6. y′=-2o$\sin(10x)\cos(10x)$