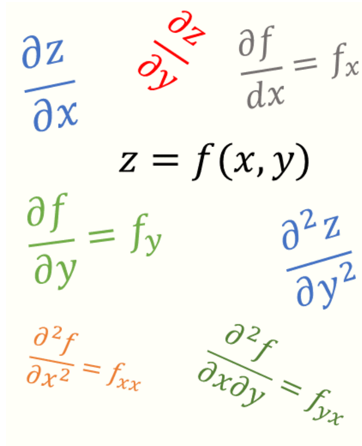
D13: Partial Differentiation



Let us suppose that we have the equation for a paraboloid with an elliptical cross-section such as $z = x^2 + 4y^2$. In this case we have a function of two independent variables, $z = f(x, y)$, and its graph is a 3-dimensional surface. We need to be able to differentiate z with respect to either x or y. If we treat one of the variables, say y, as a constant, then we can treat z as a function of just one variable, x. We can then calculate the derivative of z with respect to x. This derivative is called the partial derivative of z with respect to x and is denoted by $\frac{\partial z}{\partial x}$. If we treat x as a constant then we can treat z as a function of y and we can then calculate $\frac{\partial z}{\partial y}$ the partial derivative of z with respect to y.

Examples

1) If $z = x^2 + 4y^2$ find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2x + 0 \quad \text{[since } y \text{ is treated as a constant]}\\ &= 2x\\ \frac{\partial z}{\partial y} &= 0 + 8y \quad \text{[since } x \text{ is treated as a constant]}\\ &= 8y.\end{aligned}$$

2) If $f(x, y) = xy + 2y^2$ find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\begin{aligned}\frac{\partial f}{\partial x} &= y + 0 \quad \text{[Just as the derivative of } 2x \text{ is } x\text{,the derivative of } xy \text{ is } y\\ &\qquad\qquad \text{when } y \text{ is treated as a constant]}\\ &= y\\ \frac{\partial f}{\partial y} &= x + 4y\end{aligned}$$

3) If $f(x,y) = x^2y + y^2 \sin x$ find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = 2xy + y^2 \cos x$$
$$\frac{\partial f}{\partial y} = x^2 + 2y \sin x$$

4) If $f(x,y) = xy^2 \sin(xy)$ find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\begin{aligned}\frac{\partial f}{\partial x} &= y^2 \sin(xy) + xy^2 \times y \cos(xy)[\text{applying the product rule}]\\ &= y^2 \sin(xy) + xy^3 \cos(xy)\\ \frac{\partial f}{\partial y} &= 2xy \sin(xy) + xy^2 \times x \cos(xy)\\ &= 2xy \sin(xy) + x^2y^2 \cos(xy)\end{aligned}$$

Alternative Notation and Evaluation at a Point

An alternative notation for partial derivatives is f_x for $\frac{\partial f}{\partial x}$and f_y for $\frac{\partial f}{\partial y}$.

Partial derivatives may be evaluated at particular points: $f_x(2,1)$ refers to the value of the partial derivative of f with respect to x at the point where $x = 2$ and $y = 1$.

Example

If $f(x,y) = 2x^3y + 3y^2$ find $f_y(1,3)$

$$\begin{aligned} f(x,y) &= 2x^3y + 3y^2\\ \Rightarrow f_y &= 2x^3 + 6y\\ \Rightarrow f_y(1,3) &= 2 + 18\\ &= 20.\end{aligned}$$

Higher Order Partial Derivatives

As we have seen, a function z= $f(x,y)$ has two partial derivatives. They are $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

A function such as this will have four second order partial derivatives:

1. It can be differentiated with respect to x and then with respect to x again $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$=$\frac{\partial^2 f}{\partial x^2} = f_{xx}$
2. It can be differentiated with respect to y and then with respect to y again$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$=$\frac{\partial^2 f}{\partial y2} = f_{yy}$

3. It can be differentiated with respect to x and then with respect to y $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$
4. It can be differentiated with respect to y and then with respect to x $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y \partial x} = f_{yx}$

Example

Find all second order partial derivatives of $f(x, y) = x^2y^3 + 2x\cos y$

$$f_x = 2xy^3 + 2\cos y$$
$$\Rightarrow f_{xx} = 2y^3 \text{and } f_{xy} = 6xy^2 - 2\sin y$$
$$f_y = 3x^2y - 2x\cos y$$
$$\Rightarrow f_{yy} = 6x^2y - 2x\sin y \text{and } f_{yx} = 6xy^2 - 2\sin y$$

(Note that $f_{xy} = f_{yx}$. This will always be the case.)

Exercises

1. Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$for each of the following
   a) $z = 3x^4 + 2y^3$
   b) $z = x^2y$
   c) $z = 3xe^{2y}$
   d) $z = \ln(x^3y^5 - 2)$
   Answers
   a) $\frac{\partial z}{\partial x} = 12x^3, \frac{\partial z}{\partial y}=6y^2$
   b) $\frac{\partial z}{\partial x} = 2xy, \frac{\partial z}{\partial y} = x^2$
   c) $\frac{\partial z}{\partial x} = 3e^{2y}, \frac{\partial z}{\partial y} = 6xe^{2y}$
   d) $\frac{\partial z}{\partial x}=\frac{3x^2y^5}{x^3y^5-2}, \frac{\partial z}{\partial y} = \frac{5x^3y^4}{x^3y^5-2}$
2. Find the value of the indicated partial derivative at the given point
   a) $f(x, y) = x^4 - 4y^2$, find $f_x(2, 3)$
   b) $f(x, y) = \ln(x^2 + y^3)$, find $f_y(-1, 1)$
   Answers
   a) 32
   b) 1.5
3. Find the first and second order partial derivatives of the following
   a) $f(x, y) = x\ln(y)$
   b) $f(x, y) = x^3 + x^2y - 3xy^2 + y^3$
   c) $f(x, y) = \sin(xy)$
   d) $f(x, y) = x\cos y + ye^x$
   Answers
   a) $f_x = \ln y, f_y = \frac{x}{y}, f_{xx} = 0, f_{xy} = -\frac{x}{y^2}, f_{xy} = f_{yx} = \frac{1}{y}$
   b) $f_x = 3x^2 + 2xy - 3y^2, f_y = x^2 - 6xy + 3y^2,$

$f_{xx} = 6x + 6y, f_{yy} = -6x + 6y, f_{xy} = f_{yx} = 2x - 6y$
c) $f_x = y\cos(xy),\ f_y = x\cos(xy),\ f_{xx} = -y^2\sin(xy),$
$f_{yy} = -x^2\sin(xy),\ f_{xy} = f_{yx} = -xy\sin(xy) + \cos(xy)$
d) $f_x = \cos y + ye^x, f_y = x\sin y + e^x, f_{xx} = ye^x,$
$f_{yy} = -x\cos y, f_{xy} = f_{yx} = e^x - \sin y$