RMIT UNIVERSITY

# D12: Implicit Differentiation

$$y = f(x)$$

$$\frac{d}{dx}\sin(xy) = \frac{d}{du}(\sin u)\frac{du}{dx}$$

$$= \cos(u)\,(y + xy')$$

$$= \cos(xy)(y + xy')$$

If we are able to write an equation relating $x$ and $y$ explicitly, that is in the form $y = f(x)$, then we can find the derivative function $y = f'(x)$ or $y = \frac{dy}{dx}$ using the rules we have learned so far.

For example, if

$$y = 3x^2 - 2x + 5$$

then

$$\frac{dy}{dx} = 6x - 2.$$

But how do we differentiate expressions such as $y^5 + 3xy + x^2 - 5 = 0$ or find $\frac{d}{dx}(sin(xy))$?

In such expressions $y$ is said to be an implicit function of $x$ as we cannot rearrange the expression to the form $y = f(x)$. But we can use implicit differentiation techniques to find $\frac{dy}{dx}$ without having to solve the given equation for $y$.

Within this process the chain rule must be used whenever the function $y$ is being differentiated because it is assumed that $y$ is an unknown function of $x$.

## *The Chain Rule*

Consider $\frac{d}{dx}(y^2)$

If $y = \sin(x)$, $\frac{d}{dx}(y^2)$ becomes $\frac{d}{dx}([\sin(x)]^2)$. Applying the chain rule $\frac{dy}{dx} = 2\sin(x)\cdot\cos(x) = 2y\cdot\frac{dy}{dx}$

If $y = (4x+3)$, $\frac{d}{dx}(y^2)$ becomes $\frac{d}{dx}(4x+3)^2$. Applying the chain rule $\frac{dy}{dx} = 2(4x+3)\cdot 4 = 2y\cdot\frac{dy}{dx}$

If $y = e^x$, $\frac{d}{dx}(y^2)$ becomes $\frac{d}{dx}(e^x)^2$.

Applying the chain rule $\frac{dy}{dx} = 2\left[(e^x)\right]\cdot e^x = 2y.\frac{dy}{dx}$

And when $y$ is an unspecified function of $x$, $\frac{d}{dx}(y^2) = 2y\cdot\frac{dy}{dx}$

More generally, using the chain rule, if $u = f(y)$ then

$$\frac{du}{dx} = \frac{du}{dy}\cdot\frac{dy}{dx}. \qquad (1)$$

*Example 1*

Find $\frac{d}{dx}\left(y^2 x\right)$.

**Solution:**

Using the product rule first

$$\begin{aligned}\frac{d}{dx}\left(y^2x\right) &= y^2\frac{d}{dx}(x) + x\frac{d}{dx}(y^2)\\ &= y^2\cdot 1 + x\frac{d}{dy}\left(y^2\right)\frac{dy}{dx} \text{ using (1) above}\\ &= y^2 + x\cdot 2y\frac{dy}{dx}\\ &= y^2 + 2xy\frac{dy}{dx}.\end{aligned}$$

*Example 2*

Find $\frac{dy}{dx}$ if $y^3 = 2xy - 7$. [1]

**Solution:**

$$\begin{aligned}y^3 &= 2xy - 7\\ 3y^2y' &= 2x\cdot 1y' + 2\cdot 1\cdot y - 0\\ 3y^2y' &= 2xy' + 2y\\ 3y^2y' - 2xy' &= 2y\\ (3y^2 - 2x)y' &= 2y\\ y' &= \frac{2y}{3y^2 - 2x}.\end{aligned}$$

[1] $\frac{dy}{dx}$ may be abbreviated by $y'$. Sometimes this is helpful in reducing the amount you have to write.
In this example we use (1) on the left hand side and the product rule on the term $2xy$.

*Example 3*

Find the value of the derivative at the point $(\pi, 0)$ if $\sin(xy) = 2x$.

**Solution:**

We have

$$\sin(xy) = 2x.$$

Using the chain rule on $\sin(xy)$ and the product rule on $xy$ we obtain

$$\begin{aligned}\cos(xy)\cdot\left(xy' + y\right) &= 2\\ \text{At } (\pi,0),\ \cos(\pi\cdot 0)\cdot\left(\pi y' + 0\right) &= 2\\ (1)\cdot\left(\pi y'\right) &= 2\\ \pi y' &= 2\\ y' &= \frac{2}{\pi}.\end{aligned}$$

*Example 4*

Find the equation of the tangent line to the circle $x^2 + y^2 = 9$ at the point $(2, \sqrt{5})$.

**Solution:**

Differentiating implicitly, we have

$$\begin{aligned} 2x + 2yy' &= 0 \\ y' &= \frac{-2x}{2y} \\ &= -\frac{x}{y}. \end{aligned}$$

At the point $(2, \sqrt{5})$, $y'$ and so the gradient $m$ of the tangent is

$$m = -\frac{2}{\sqrt{5}}.$$

Hence the equation of the tangent is[2]

$$\begin{aligned} y - \sqrt{5} &= -\frac{2}{\sqrt{5}}(x-2) \\ y &= -\frac{2}{\sqrt{5}}x + \frac{4}{\sqrt{5}} + \sqrt{5} \\ &= -\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}(4+5) \\ &= -\frac{2}{\sqrt{5}}x + \frac{9}{\sqrt{5}}. \end{aligned}$$

[2] Remember that the equation of a straight line with gradient $m$ through the point $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1).$$

## *Exercises*

1. Find $\frac{d}{dx}\left(\frac{x}{y}\right)$. Hint: use quotient rule.
   Answer
   $\frac{y-xy'}{y^2}$.
2. Find $\frac{d}{dx}\left(\frac{x+y}{x-y}\right)$.
   Answer
   $\frac{2xy'}{(x-y)^2}$.
3. Find the value of $\frac{dy}{dx}$ at the point (1, 2) if $x^2 + y = 7 - 2xy$.
   Answer
   $-2$.
4. Find $\frac{dy}{dx}$ if $e^x - \sin(y) = x$ .
   Answer
   $\frac{e^x - 1}{cosy}$.
5. Find the equation of the tangent line to the circle $x^2 + y^2 = 4$ at the point $(1, \sqrt{3})$.
   Answer
   $y = -\frac{1}{\sqrt{3}}x + \frac{4}{\sqrt{3}}$.