D10: *Rates of Change*

If there is a relationship between two or more variables, for example, area and radius of a circle (A=πr^2), or length of a side and volume of a cube ($V = l^3$), or days since first case and number of people with an infectious disease then there will also be a relationship between the rates at which the variables change. If y is a function of x, that is $y = f(x)$, then $\frac{dy}{dx} = f'(x)$.

We can use differentiation to find the function that defines the rate of change between variables

Image from Pixabay.

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$$
$$\text{and } V = l^3 \Rightarrow \frac{dV}{dl} = 3l^2$$

The chain rule can be used to find rates of change with respect to time:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\text{and } V = l^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt}$$
$$\frac{dV}{dt} = 3l^2 \frac{dl}{dt}$$

Examples

1. A balloon has a small hole and its volume V in cubic centimeters after t seconds is $V = 66 - 10t - 0.01t^2$, $t > 0$. Find the rate of change

of volume after 10 seconds.

$$\begin{aligned} V &= 66 - 10t - 0.01t^2 \\ \frac{dV}{dt} &= -10 - 0.02t \\ \text{When } t = 10, \frac{dV}{dt} &= -10 - 0.02\,(10) \\ &= -10.2\,cm^3/s. \end{aligned}$$

The volume of the balloon is decreasing at a rate of $10.2cm^3/s$.

2. The pressure P, of a given mass of gas kept at constant temperature, and its volume V are connected by the equation $PV = 500$. Find $\frac{dP}{dV}$ when $V = 20$.

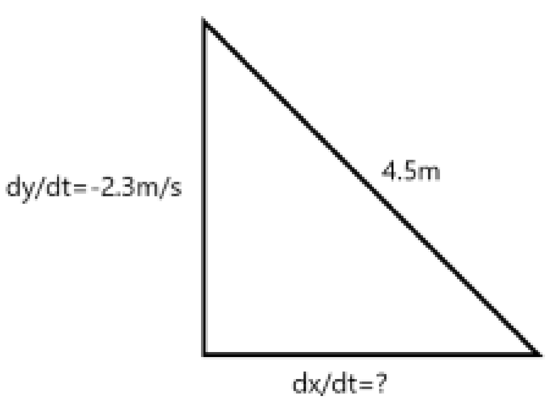
$$\begin{aligned} PV = 500 \Rightarrow P &= \frac{500}{V} \\ P &= 500V^{-1} \\ \text{Then } \frac{dP}{dV} &= -500V^{-2} \\ V = 20 \Rightarrow \frac{dP}{dV} &= -500(20)^{-2} \\ &= -1.25. \end{aligned}$$

The rate of change of pressure with respect to volume is -1.25.

3. A ladder 4.5m long ladder is sliding down a vertical wall with the top descending at a rate of $2.3\,m/s$. How fast is the bottom of the ladder moving along the ground when the bottom is 3 meters from the wall?

A diagram reveals that the information in the question is described by Pythagoras's theorem:[1]

[1] Pythagoras's theorem: $a^2 + b^2 = c^2$ where c is the hypotenuse and a and b the shorter sides of a right triangle.



If y is the height the ladder reaches up the wall and x is the distance of the bottom of the ladder from the wall then

$$
\begin{aligned}
x^2 + y^2 &= 4.5^2 \\
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \quad \text{[differentiating implicitly]} \\
2 \times 3 \times \frac{dx}{dt} + 2 \times 3.35 \times (-2.3) &= 0 \quad [a = 3, c = 4.5 \Rightarrow b = 3.35] \\
6\frac{dx}{dt} - 15.41 &= 0 \\
\Rightarrow \frac{dx}{dt} &= 2.57
\end{aligned}
$$

The ladder is moving along the ground at a speed of 2.57 m/s.

Exercise

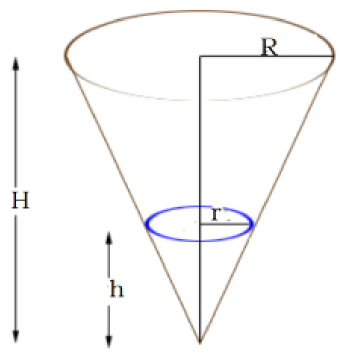
1. The radius of a spherical balloon is increasing at a rate of 3cm/min. At what rate is the volume increasing when the radius is 5cm?
   Answer: $300\pi \approx 942 cm^3/min$

2. If the displacement of an object from a starting point is given by $s(t) = \sin(t) - 2\cos(t)$ find the velocity when $t = 1$. Hint: $v(t) = s'(t) = \frac{ds}{dt}$
   Answer: 2.22

3. The function $n(t) = 200t - 100\sqrt{t}$ describes the spread of a virus where t is the number of days since the initial infection and n is the number of people infected. Find the rate at which n is increasing at the instant when $t = 4$.
   Answer: 175 people per day.

4. If $y = (x - \frac{1}{x})^2$ find $\frac{dx}{dt}$ when $x = 2$, given $\frac{dy}{dt} = 1$.
   Answer:4/15

5. A hollow right circular cone is held vertex downwards beneath a tap leaking at the rate of 2 cm^3/s. Find the rate of rise of water level when the depth is 6 cm given that the height of the cone is 18 cm and its radius is 12 cm. (Hint: Use the properties of similar triangles to find a relationship between radius and height.)



Answer: $\frac{1}{8\pi}\,cm/s \approx 0.04\,cm/s$